9^{th} MATHEMATICAL DUEL

PROBLEMS AND SOLUTIONS

(Chorzów – March 20, 2001)

A–I–1

Prove that the number

$$L = m^{21}n^3 - m^7n - m^3n^{21} + mn^7$$

is divisible by 42 for any integers m, n.

Solution. First we will prove that L is divisible by the prime 7. From Fermat's theorem it follows that the congruence $a^7 \equiv a \pmod 7$ is satisfied for any integer a. Thus

$$L \equiv m^3n^3 - mn - m^3n^3 + mn = 0 \pmod 7 .$$

Analogously the congruence $a^3 \equiv a \pmod 3$, which is true by Fermat's theorem for any integer a and the prime 3, yields

$$L \equiv mn - mn - mn + mn = 0 \pmod 3 .$$

The proof of the divisibility L by the number 2 is obvious for reasons of parity.

Therefore the number L is divisible by the product of the primes 2, 3 and 7 ($2 \cdot 3 \cdot 7 = 42$).

A–I–2

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equation

$$f(f(x+y)) = f(x) + y$$

is satisfied for arbitrary $x, y \in \mathbb{R}$.

Solution. We can change x and y and obtain

$$f(f(y+x)) = f(y) + x \qquad \text{for all} \quad x, y \in \mathbb{R} .$$

¿From this functional equation immediately follows

$$f(x) + y = f(y) + x \qquad \text{for all} \quad x, y \in \mathbb{R} .$$

Taking $y = 0$ we get

$$f(x) = x + f(0) .$$

For the left side of the given equation we get the expression

$$f(f(x+y)) = f(x+y+f(0)) = (x+y+f(0)) + f(0) = x+y+2f(0)$$

and for its the right side it holds

$$f(x)+y = (x+f(0))+y = x+y+f(0)\,.$$

¿From last two conditions we get $f(0)=0$ and therofore $f(x)=x$ for all $x\in\mathbb{R}$. Further we see (after checking) that the unique solution of the given equation is the function $f(x)=x$ for all $x\in\mathbb{R}$.

A–I–3

Construct a quadrilateral inscribed in the circle with the radius 3 cm with two opposite sides of the lengths 2 cm and 4 cm and with maximum area.

Solution. For the lengths of sides of the cyclic quadrilateral $ABCD$ we can consider without loss of generality $|AB| = 4\,\mathrm{cm}$ and $|CD| = 2\,\mathrm{cm}$. From the adjoining figure we can see that the equalities

$$|\angle DAC| = |\angle DBC| = \omega \qquad \text{and} \qquad |\angle ACB| = |\angle ADB| = \vartheta$$

hold.

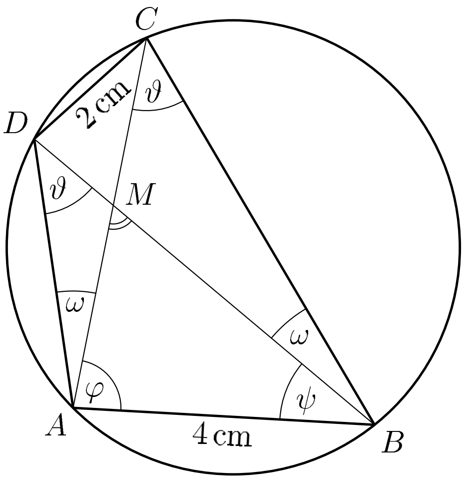

Figure 1

For the given lengths of the opposite sides AB and CD of the cyclic quadrilateral $ABCD$, φ and ψ are constant. Let M be the point of intersection of the diagonals AC and BD. Because ω and ϑ are constant $|\angle AMB| = \omega+\vartheta$ is constant too. We note $|\angle BAC| = \varphi$ and $|\angle DBA| = \psi$. For the lengths of the diagonals AC and BD we have

$$|AC| = 2r\sin(\omega+\varphi)\,, \qquad |BD| = 2r\sin(\omega+\psi)\,.$$

The area P of the cyclic quadrilateral $ABCD$ is given by the formula $P = \frac{1}{2}\cdot|AC|\cdot|BD|\cdot\sin|\angle AMB| = \frac{1}{2}\cdot 4r^2\cdot\sin|\angle AMB|\cdot\sin(\omega+\varphi)\cdot\sin(\omega+\psi)\,.$ Because the product $\frac{1}{2}\cdot 4r^2\cdot\sin|\angle AMB|$ is constant we will maximize

$$\sin(\omega+\varphi)\cdot\sin(\omega+\psi)\,.$$

Therefore

$$\sin(\omega+\varphi) = \sin(\omega+\psi) = \frac{1}{2}\Big(\cos(\varphi-\psi)-\cos(\varphi+\psi+2\omega)\Big) \le \frac{1}{2}\Big(1-\cos(\varphi+\psi+2\omega)\Big)\,.$$

The equality holds if and only if $\varphi = \psi$. This conclusion is true in the case when $ABCD$ is an isosceles trapezium with bases AB and CD.

The construction of the quadrilateral $ABCD$ is then evident.

Remark. Another solution can use the Brahmagupta's formula for the area of a cyclic quadrilateral.

A–I–4

Let a, b, c be the lengths of the sides of a triangle. Prove that the inequality

$$3a^2 + 2bc \ge 2ab + 2ac$$

is fulfilled.

Solution. We have

$$\begin{aligned}
& 3a^2 + 2bc > 2ab + 2ac \\
\iff\quad & 3a^2 + 2bc - 2ab - 2ac > 0 \\
\iff\quad & a^2 - b^2 + 2bc - c^2 + a^2 - 2ab + b^2 + a^2 - 2ac + c^2 > 0 \\
\iff\quad & a^2 - (b-c)^2 + (a-b)^2 + (a-c)^2 > 0 \\
\iff\quad & (a-b+c)(a+b-c) + (a-b)^2 + (a-c)^2 > 0\,.
\end{aligned}$$

The expressions $(a-b)^2$ and $(a-c)^2$ are certainly non-negative, and due to the triangle inequality, we have

$$a + c > b \quad \text{and} \quad a + b > c\,.$$

Therefore

$$(a-b+c)(a+b-c) > 0$$

and the result follows.

A–T–1

Prove that for arbitrary positive real numbers a, b, c the inequality

$$\frac{1}{ab(a+b)} + \frac{1}{bc(b+c)} + \frac{1}{ca(c+a)} \geq \frac{9}{2(a^3+b^3+c^3)}$$

is satisfied. When does equality hold?

Solution. We consider the inequality

$$a^3 + b^3 \geq ab(a+b) = a^2b + ab^2$$

which is true for arbitrary positive real numbers a, b. We rewrite this inequality in the form

$$\frac{1}{ab(a+b)} \geq \frac{1}{a^3+b^3}\,.$$

Similarly we have (for all positive real numbers a, b, c)

$$\frac{1}{bc(b+c)} \geq \frac{1}{b^3+c^3}\,,$$

$$\frac{1}{ca(c+a)} \geq \frac{1}{c^3+a^3}\,.$$

The sum of the last three inequalities yields

$$\frac{1}{ab(a+b)} + \frac{1}{bc(b+c)} + \frac{1}{ca(c+a)} \geq \frac{1}{a^3+b^3} + \frac{1}{b^3+c^3} + \frac{1}{c^3+a^3}\,.$$

Applying the arithmetic-harmonic mean inequality for the right side of the last inequality we get

$$\frac{1}{3}\left(\frac{1}{a^3+b^3} + \frac{1}{b^3+c^3} + \frac{1}{c^3+a^3}\right) \geq \frac{3}{(a^3+b^3)+(b^3+c^3)+(c^3+a^3)}\,.$$

After easy manipulation we obtain immediately

$$\frac{1}{a^3+b^3} + \frac{1}{b^3+c^3} + \frac{1}{c^3+a^3} \geq \frac{9}{2(a^3+b^3+c^3)}\,.$$

The equality holds if and only if $a = b = c$.

A–T–2

Let ABC be a triangle. Equilateral triangles ABD, BCE and CAF are drawn outside of ABC. Prove that the midpoints of these three triangles are the vertices of an equilateral triangle.

Solution. Let ABD, BCE and CAF be the equilateral triangles which are drawn outside the given triangle ABC. It is easy to see that all their circumcircles intersect at a common point O (Fig. 2) for which it holds

$$|\angle AOB| = |\angle BOC| = |\angle COA| = 120^\circ .$$

Let X, Y, Z be the midpoints of the equilateral triangles BCE, CAF, ABD, respectively. These points are the centres of circumcircles about the triangles ABD, BCE and CAF too. Therefore the segment XY is perpendicular to chord CO, YZ is perpendicular to chord AO. Hence the angles XYZ and COA are supplementary. Therefore it is $|\angle XYZ| = 60^\circ$. Analogously we can prove that $|\angle YZX| = |\angle ZXY| = 60^\circ$. So the triangle XYZ is equiangular and hence equilateral.

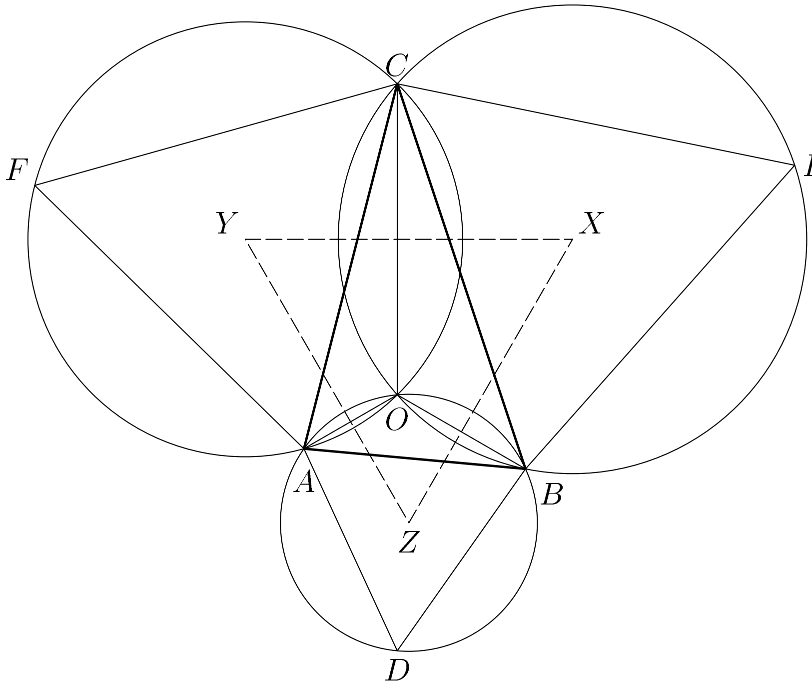

Figure 2

A–T–3

Is it possible for three different positive integers x, y, z to exist between two successive perfect squares, such that one is the geometric mean of the other two? In other words, is it possible that $n^2 < a, b, c < (n+1)^2$ and $c = \sqrt{ab}$ hold with a, b, c all different? If so, give an example. If not, prove why this is not possible.

Solution. It is not possible. In order to see this, we assume without loss of generality $n^2 < a < b < (n+1)^2$. We can write $a = \overline{a}^2 \cdot r$, where r does not contain any prime factors twice, i.e. r is square-free. Since $ab = c^2$ is a perfect square, we must have $b = \overline{b}^2 \cdot r$. It therefore follows that

$$n^2 < a < b < (n+1)^2 \iff n^2 < \overline{a}^2 \cdot c < \overline{b} \cdot r < (n+1)^2$$

$$\iff \left(\frac{n}{\sqrt{r}}\right)^2 < \overline{a}^2 < \overline{b}^2 < \left(\frac{n+1}{\sqrt{r}}\right)^2$$

$$\iff \frac{n}{\sqrt{r}} < \overline{a} < \overline{b} < \frac{n+1}{\sqrt{r}} = \frac{n}{\sqrt{r}} + \frac{1}{\sqrt{r}} \leq \frac{n}{\sqrt{r}} + 1\,.$$

Since $r \geq 1$, we have $\frac{1}{\sqrt{r}} \geq 1$, and so $\overline{a}$ and $\overline{b}$ must be different integers between $\frac{n}{\sqrt{r}}$ and $\frac{n}{\sqrt{r}} + 1$, which contradicts the fact that there can only be at most one integer in this interval.

B–I–1

Determine all integers x such that

$$f(x) = \frac{x^3 - 2x^2 - x + 6}{x^2 - 3}$$

is an integer.

Solution. Since $x^2 \neq 3$ for $x \in \mathbb{Z}$, we see that

$$\frac{x^3 - 3x^2 - x + 6}{x^2 - 3} = \frac{x^3 - 3x^2 - 3x + 6}{x^2 - 3} + \frac{2x}{x^2 - 3} =$$

$$= \frac{x^3 - 3x^2 - x + 6}{x^2 - 3} + \frac{2x}{x^2 - 3} = x - 2 + \frac{2x}{x^2 - 3}$$

is an integer if and only if $(x^2 - 3)|2x$. This can only be the case if $|x^2 - 3| \leq |2x|$ i.e. for $x \in \{-3, -2, -1, 0, 1, 2, 3\}$. Checking these numbers, we see that

$$f(-3) = -6, \quad f(-2) = -8, \quad f(-1) = -2, \quad f(0) = -2, \quad f(1) = -2,$$

$$f(2) = 4 \quad \text{and} \quad f(3) = 2$$

are indeed all integers, and the x for which $f(x)$ is an integer are the elements of $\{-3, -2, -1, 0, 1, 2, 3\}$.

B–I–2

A convex quadrilateral in the plane is given. Prove that the lines passing through the midpoints of its opposite sides are perpendicular if and only if the diagonals of the given quadrilateral are of the same length.

Solution.

a) First let $|AC| = |BD|$.

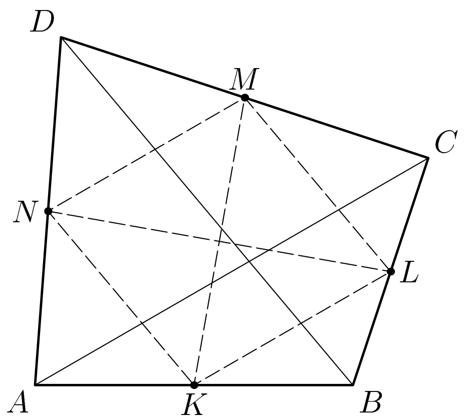

Figure 3

From Fig. 3 we can see that KL and MN are congruent segments which are parallel to the diagonal AC. Further we have

$$|KL| = |MN| = \tfrac{1}{2}|AC|\,.$$

Similarly

$$|NK| = |LM| = \tfrac{1}{2}|BD|\,.$$

According to $|AC| = |BD|$ we get

$$|KL| = |LM| = |MN| = |NK|\,.$$

Therefore the quadrilateral $KLMN$ is either a square or a parallelogram with sides of the same length. Its diagonals KM and LN are therefore perpendicular.

b) Let $KM \perp LN$.

The diagonals KM and LN of the (Varignon's) quadrilateral $KLMN$ are perpendicular in this case. Because $KLMN$ is a certainly parallelogram, its diagonals are divided into two equal parts. This implies that the quadrilateral $KLMN$ is either square or a parallelogram with all sides of the same length. It follows that $|AC| = |BD|$ holds.

B–I–3

Four real numbers a, b, c, d are given such that

$$b - a = c - b = d - c\,.$$

The sum of all these numbers is 2 and the sum of their cubes is $\frac{4}{3}$. Determine the numbers.

Solution. Denote $2r = b - a = c - b = d - c$ and $m = \frac{b+c}{2}$. Then

$$c = \frac{b+c}{2} + \frac{c-b}{2} = m + r, \quad b = \frac{b+c}{2} - \frac{c-b}{2} = m - r \quad d = m + 3r, \quad a = m - 3r.$$

Since

$$a + b + c + d = 2 \iff (m - 3c) + (m - c) + (m + c) + (m + 3c) = 2$$

$$\iff 4m = 2 \iff m = \frac{1}{2}$$

and

$$(m-3c)^3 + (m-r)^3 + (m+r)^3 + (m+3r)^3 = \frac{4}{3} \iff 4m^3 + 54mc^2 + 6mc^2 = \frac{4}{3}$$

$$\iff 4m^3 + 60mc^2 = \frac{4}{3} \implies \frac{1}{2} + 30c^2 = \frac{4}{3} \implies c^2 = \frac{1}{36} \iff c = \frac{1}{6},$$

the numbers are

$$0\,,\ \frac{1}{3}\,,\ \frac{2}{3}\,,\ 1\,.$$

B–I–4

Determine how many triples (a, b, c) of positive integers satisfy the following system of inequalities

$$\frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} + \frac{c^2}{c^2 + 2ab} \le 1\,,$$
$$a^2 + b^2 + c^2 \le 2001\,.$$

Solution. Using the following well-known inequalities

$$a^2 + b^2 \ge 2ab\,, \qquad b^2 + c^2 \ge 2bc\,, \qquad c^2 + a^2 \ge 2ca\,,$$

which is true for any positive integers a, b, c, we obtain

$$\frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} + \frac{c^2}{c^2 + 2ab} \ge \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2} = 1\,.$$

From the first inequality of the given system of inequalities it further follows that

$$\frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} + \frac{c^2}{c^2 + 2ab} = 1\,.$$

This equality implies $a = b = c$. From the second inequality of the given system we therefore obtain the condition $3a^2 \le 2001$, i.e. $a^2 \le 667$. This inequality is fulfilled for any integer a from the set $\{0, 1, \ldots, 25\}$.

The number of integer solutions of the given system of inequalities is 25. They are all triples in the form (k, k, k), where

$$k \in \{1, 2, \ldots, 25\}\,.$$

B–T–1

Solve the following system of equations

$$\begin{aligned} x^2y + y^2z &= 254\,, \\ y^2z + z^2x &= 264\,, \\ z^2x + x^2y &= 6 \end{aligned}$$

in real numbers.

Solution. Adding all these equations yields

$$2(x^2y + y^2z + z^2x) = 524$$

or

(1) $$x^2y + y^2z + z^2x = 262\,.$$

Subtracting each of the three original equations from (1) yields the equivalent system of equations

(2) $$z^2x = 8\,,$$

(3) $$x^2y = -2\,,$$

(4) $$y^2z = 256\,.$$

Multiplying these three equations yields

$$(xyz)^3 = -4096$$

or

(5) $$xyz = -16\,.$$

Dividing (2), (3) and (4) by (5) yields

$$\frac{z}{y} = -\frac{1}{2}\,, \quad \frac{x}{z} = \frac{1}{8}\,, \quad \frac{y}{x} = 16,$$

and it therefore follows that

$$x^3 = z^2x \cdot \left(\tfrac{x}{z}\right)^2 = 8 \cdot \left(\tfrac{1}{8}\right) = \tfrac{1}{8} \quad \Longrightarrow \quad x = \tfrac{1}{2}\,,$$

$$y^3 = x^2y \cdot \left(\tfrac{y}{x}\right)^2 = -2 \cdot 16^2 = -512 \quad \Longrightarrow \quad y = -8 \text{ and}$$

$$z^3 = y^2 z \cdot \left(\tfrac{z}{y}\right)^2 = 256 \cdot \left(-\tfrac{1}{2}\right)^2 = 64 \quad \Longrightarrow \quad z = 4\,.$$

B–T–2

The square $ABCD$ with sides of length 4 cm is given. Let K and L be the midpoints of its sides BC and CD, respectively. Determine the area of the quadrilateral whose sides lie on the lines AK, AC, BL and BD.

Solution.

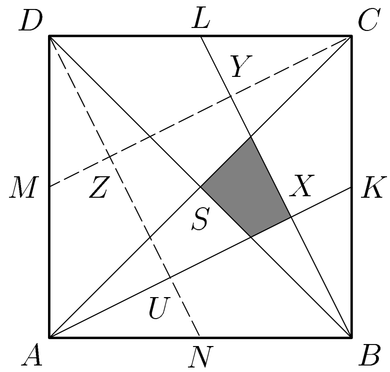

Figure 4

According to the central symmetry with the centre in the point of intersection S of the diagonals AC and BD of the given square $ABCD$ we consider the midpoints M, N of the segments DA, AB, respectively. Let X, Y, Z, U be the points of intersection of the lines AK with BL, BL with CM, CM with DN, DN with AK, respectively (see Fig. 4). By rotation with centre S and angle 90° it is easy to see that the points X, Y, Z and U are the vertices of a square. Further, we can compute the area of the square $XYZU$ which is $\frac{16}{5}\,\mathrm{cm}^2$. The area of the quadrilateral whose sides lie on the lines AK, AC, BL and BD is therefore one quarter of the area of the square $XYZU$ i.e. $\frac{4}{5}\,\mathrm{cm}^2$.

B–T–3

The quadratic equation $ax^2 + bx + c = 0$ without real roots is given such that $a + b + c < 0$. Determine the sign of the coefficient c.

Solution. Since the equation $ax^2 + bx + c = 0$ has no real roots, we have $b^2 - 4ac < 0$ and it follows that $ac > \frac{b^2}{4}$.

Let us now prove that the product of c and $a + b + c$ is positive:

$$c(a + b + c) = ac + bc + c^2 > \frac{b^2}{4} + bc + c^2 = \left(\frac{b}{2} + c\right)^2 \geq 0\,.$$

Thus $c < 0$, because $c(a + b + c) > 0$ and $a + b + c < 0$.

C–I–1

Solve in integers the equation

$$7(mx + 3) = 3(2mx + 9)\,,$$

where m is a given integer. Consider all possibilities for m.

Solution. The given equation is equivalent to the equation

$$7mx + 21 = 6mx + 27$$

or

$$mx = 6\,.$$

For $m \neq 0$ we have $x = \frac{6}{m}$ and $m \in \{-6, -3, -2, -1, 1, 2, 3, 6\}$. For $m = 0$ the above equation has no solution.

C–I–2

The rectangle $ABCD$ with sides of the lengths a, b $(a > b)$ is given. Consider the rectangle $ACEF$ such that the vertex D lies on the segment EF. The sides CD, DA divide the rectangle $ACEF$ into three triangles whose areas are in the ratio $1 : 2 : 3$. Determine the ratio $a : b$.

Solution. The sides CD and DA of the rectangle $ABCD$ divide the rectangle $ACEF$ into three similar right-angled triangles ACD, CDE, DAF (Fig. 5) with the areas P_1, P_2, P_3 respectively, such that the ratio $P_1 : P_2 : P_3 = 3 : 2 : 1$ holds.

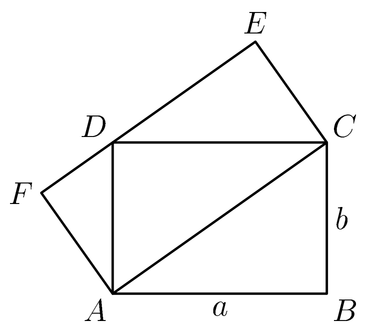

Figure 5

For the lengths of their hypotenuses AC, CD, DA we therefore have

$$|AC| : |CD| : |DA| = \sqrt{3} : \sqrt{2} : 1$$

and so

$$a : b = |AB| : |BC| = |CD| : |DA| = \sqrt{2} : 1\,.$$

C–I–3

Determine all pairs (m, n) of positive integers such that the conditions

$$\langle m, n\rangle = 125 \qquad \text{and} \qquad [m, n] = 10\,000$$

are satisfied; $\langle m, n\rangle$ and $[m, n]$ denote greatest common divisor and least common multiple of positive integers m, n, respectively.

Solution. Observe that $\langle m, n\rangle = 5^3$ and $[m, n] = 2^4 \cdot 5^4$. Since we have here exclusively the powers of the numbers 2 and 5 and $\langle m, n\rangle = 5^3$, the following four cases are only possible

(i) $m = 5^3 \cdot 5\,, n = 5^3 \cdot 2^4$ and solution (625,2000),

(ii) $m = 5^3 \cdot 2^4\,, n = 5^3 \cdot 5$ and solution (2000,625),

(iii) $m = 5^3 \cdot 2^4 \cdot 5\,, n = 5^3$ and solution (10000,125),

(iv) $m = 5^3\,, n = 5^3 \cdot 2^4 \cdot 5$ and solution (125,10000),

and there are no other solutions.

C–I–4

The right-angled triangle ABC with right angle at C is given. Let r be the inradius of the triangle ABC. Prove that the equality

$$a + b = c + 2r$$

holds.

Solution. Let ABC be a right-angled triangle (see Fig. 6)

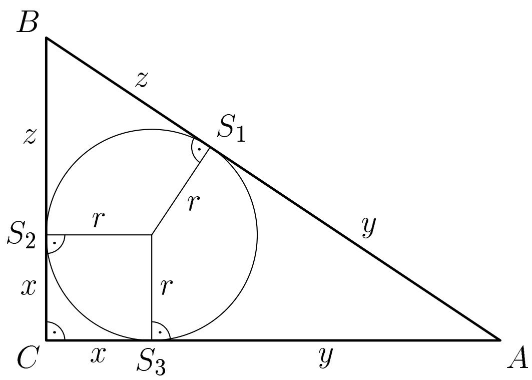

Figure 6

If S stands for the center of the circle inscribed in the triangle ABC and S_1, S_2, S_3 denote the tangent points of the circle inscribed in the triangle with the sides AB, BC, CA, respectively, then

$$|S_3A| = |AS_1| = y\,,$$
$$|S_1B| = |BS_2| = z\,,$$

$$|S_2C| = |CS_3| = x\,.$$

Let us observe that CS_3SS_2 is a square, so that $x = r$. Note that

$$a + b = z + x + x + y = y + z + 2x = y + z + 2r\,,$$
$$c + 2r = z + y + 2r\,,$$

which proves the required equalities.

C–T–1

Determine all triples (x, y, z) of prime numbers such that the conditions

$$x < y < z\,, \qquad x + y + z = 77$$

hold.

Solution. The prime numbers less then 77 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73. Among them we will seeking the solutions of the equation satisfying $x < y < z$.

Let us consider the following possible cases

(i) $x = 2$, then $y + z = 75$ and there is no solution,

(ii) $x = 3$, then $y + z = 74$, and we have the solutions$(3, 7, 67)$, $(3, 13, 61)$, $(3, 31, 43)$,

(iii) $x = 5$, then $y+z = 72$, and we have the solutions $(5, 11, 61)$, $(5, 13, 59)$, $(5, 19, 53)$, $(5, 29, 43)$, $(5, 31, 41)$,

(iv) $x = 7$, then $y+z = 70$, and we have the solutions $(7, 11, 59)$, $(7, 17, 53)$, $(7, 23, 47)$, $(7, 29, 41)$,

(v) $x = 11$, then $y + z = 66$, and we have the solutions $(11, 13, 53)$, $(11, 19, 47)$, $(11, 23, 43)$, $(11, 29, 37)$,

(vi) $x = 13$, then $y + z = 64$, and we have the solutions $(13, 17, 47)$, $(13, 23, 41)$,

(vii) $x = 17$, then $y + z = 60$, and we have the solutions $(17, 19, 41)$, $(17, 23, 37)$, $(17, 29, 31)$,

(viii) $x = 19$, then $y + z = 58$, and there is no solution,

(ix) $x = 23$, then $y + z = 54$, and there is no solution,

(x) $x > 23$, then sum $y+z > 68$ and there is no solution since $x+y+z > 91$.

C–T–2

An A4 sheet of paper is a rectangle with area $\frac{1}{16}$ m^2 whose sides are in the ratio $1 : \sqrt{2}$. One such A4 sheet is placed on another such that have a common diagonal, but are not identical. Determine the area of the resulting octagon.

Solution. Let $ABCDEFGH$ be the octagon resulting from placing the rectangle $ABEF$ on the rectangle $ADEH$. AE is the common diagonal of the rectangles. The mid-point M of AE is the common mid-point of the two rectangles. For reasons of symmetry, $ACEG$ is a rhombus, and the area of the octagon is equal to the sum of the areas of the two rectangles, minus the area of the rhombus, which is the overlap of the rectangles. The area of the rhombus is 4 times the area of ΔACM. Since $AM \perp CM$ and $\angle DAE = \angle CAM$, the triangles ΔACM and ΔADE are similar. If $DE = a$, we have $AD = a\sqrt{2}$, and therefore $AE = a\sqrt{3}$ and $AM = \frac{a\sqrt{3}}{2} : a\sqrt{2}$, the areas of the corresponding triangles fulfill the ratio

$$A_{\Delta ACM} : A_{\Delta ADE} = \left(\frac{\sqrt{3}}{2}\right)^2 : (\sqrt{2})^2 ,$$

and we have

$$A_{\Delta ACM} = A_{\Delta ADE} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{16} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{256} .$$

It therefore follows that the area of the octagon equals

$$A_{ABEF} + A_{ADEH} - 4 \cdot A_{\Delta ACM} = \frac{1}{16} + \frac{1}{16} - 4 \cdot \frac{3}{256} = \frac{5}{64} \text{ m}^2 .$$

C–T–3

Prove that for each non-negative real number a the inequality

$$(a^3 + a^2 - a - 1)^2 - (a^3 - a^2 - a + 1)^2 \geq 0$$

is true. When does equality hold?

Solution. Observe that

$$\begin{gathered}(a^3 + a^2 - a - 1)^2 - (a^3 - a^2 - a + 1)^2 = \\ = (a^3 + a^2 - a - 1 - a^3 + a^2 + a - 1)(a^3 + a^2 - a - 1 + a^3 - a^2 - a + 1) = \\ = (2a^2 - 2)(2a^3 - 2a) = 4a(a^2 - 1)^2 ,\end{gathered}$$

and the latter expression is non–negative for all non–negative numbers.

Equality holds for $a = 0$ or $a^2 - 1 = 0$, i.e. $a = 1$.

The competition presented in this paper was carried out for the ninth time this year. Beside the original idea of a friendly competition and the obvious mathematical benefits of such an activity for the students involved, the competition has also come to help foster understanding between cultures that, though geographically close and historically connected, were practically inaccessible to another until as recently as twenty years ago. The following paragraphs are intended to present this competition to an international audience.

The original inspiration for this activity came from several international mathematics competitions that were already well established, such as the "Baltic Way" or the Austrian-Polish Mathematics Competition. Due to the connections between Jaroslav Švrček and Józef Kalinowski and the proximity of the Copernikus Gymnázium in Bílovec (Czech Republic) and the Liceum Juliusz Słowacki in Chorzów (Poland), the idea of an international competition between students of these two schools was born, and the first such competition was held in 1993 in Bílovec. The concept of extending the competition to another school was first talked about at the WFNMC conference in 1994 in Pravetz (namely with Robert Geretschläger), and since 1997 BRG Keplerstrasse from Graz (Austria) has also been taking part. The competition now takes place alternately in Bílovec, Chorzów and Graz.

The competition is divided into three categories. There is a junior division (category C) for students in grade 8 (or younger), an intermediate division (category B) for students in grades 9 or 10, and a senior division (category A) for students in grade 11 or 12. Typically, four students from each school in each of the divisions come together for a competition (making 36 students in total). The students write an individual competition comprising 4 olympiad-style problems to be solved in 120 minutes, and a team competition comprising three problems to be solved in 90 minutes. These two competitions are completely independent of one another, and yield separate results. While the individual competition is written in supervised silence, the team competition sees one team from each school (in different divisions, of course) placed together in a room with no adult supervision. The students spend the 90 minutes attempting to formulate one common group answer to each problem, and only one answer sheet per group is accepted at the end.

The students were originally given the problems to solve in their own language (Czech, Polish or German), but for ease of organization all now receive the problems in English. They all write their answers in their own language, however. While this makes for quite some interesting language mixes during the social parts of the contest (with English often helping out as a common minimal communication tool), the multitude of languages has not been a problem for the contest itself. Specifically, the problems committee always finds ways to communicate.

The competition is organized to run four days altogether. The first day is a day of traveling. Since the train trip from Graz to Katowice (near Chorzów) takes only about $8\frac{1}{2}$ hours and Ostrava (near Bílovec) is right on the direct line, the trip is quite comfortable. The students have some time left to get used to their new surroundings, and hopefully to start making new friends.

The second day is the day of the actual competition, with the evening typically spent reading the papers, writing lists, and so on.

The results, though known to the committee, are kept secret from the competitors through-out the third day, which is spent on a group excursion. This year for instance, there was a bus trip to the famous salt mines of Wieliczka near Kraków.

Finally, the fourth day sees closing ceremonies including the handing out of prizes and diplomas, after which the guest teams head for the train station and home.