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MATHEMATICAL DUEL ’10

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Tato publikace byla připravena v rámci projektu OPVK reg. číslo CZ.1.07/1.2.12/01.0027 Zkvalitnění přípravy matematických talentů základních a středních škol Olomouckého kraje (PMT). Tento projekt je spolufinancován Evropským sociálním fondem a státním rozpočtem České republiky.

1. vydání

Preface

The 18th International Mathematical Duel was held on March 9th–12th, 2010 in Chorzów. Five school-teams from Austria, Czech Republic, Poland and Romania took part in this traditional mathematical competition, namely from BRG Kepler Graz, GMK Bílovec, GJŠ Přerov, I LO Chorzów and for the first time one team from CNC Ploieşti (Romania) as guests.

As usual the competition was provided in the three categories (A—contestants of the last two years, B—contestants of the 5th and 6th years, and C—contestants of the 3rd and 4th years of eight-year grammar school). Twelve contestants (more precisely 4 in any category) of any school took part in this competition, i.e. 60 contestants in total.

This booklet contains all problems with solutions and results of the 18th International Mathematical Duel from the year 2010.

Authors

Problems

Category A (Individual Competition)

A–I–1

Determine all triples of mutually distinct real numbers a, b, c such that the cubic equation

$$x^3 + abx^2 + bcx + ca = 0$$

with unknown x has three real roots a, b, c.

A–I–2

We are given bases $|AB| = 23$ and $|CD| = 5$ of a trapezoid $ABCD$ with diagonals $|AC| = 25$ and $|BD| = 17$. Determine the lengths of its sides BC and AD.

A–I–3

We are given a circle c_1 and points X and Y on c_1. Let XY be a diameter of a second circle c_2. We choose a point P on the greater arc XY on c_1 and a point Q on c_2 such that the quadrilateral $PXQY$ is convex and $PX \parallel QY$. Prove that a measure of the angle PYQ is independent of the choice of P (if an appropriate Q exists).

A–I–4

Solve in real numbers the following system of equations

$$\sqrt{\sqrt{x} + 2} = y - 2,$$
$$\sqrt{\sqrt{y} + 2} = x - 2.$$

Category A (Team Competition)

A–T–1

We are given two real numbers x and y ($x \neq y$) such that $x^4 + 5x^3 = y$ and $y^3 + 5x^2 = 1$ hold. Prove that the equality $x^3 + x^2y + xy^2 = -1$ holds.

A–T–2

Let us consider a unit square $ABCD$. On its sides BC and CD determine points E and F, respectively, such that $|BE| = |DF|$ and the triangles ABE and AEF have the same perimeter.

A–T–3

Determine all pairs (m, n) of integers such that the equation

$$4^n = 1899 + m^3$$

is fulfilled.

Category B (Individual Competition)

B–I–1

How many positive integers divisible by 29 in the form $\overline{abcabcabc}$ exist?

B–I–2

Determine all pairs (a, b) of positive integers such that the equation

$$4^a = b^2 + 7$$

is fulfilled.

B–I–3

We are given two line segments AB and CD in the plane. Find all points V in this plane such that the triangles ABV and CDV have the same area.

B–I–4

Determine all polynomials $P(x)$ with real coefficients and all real number q such that the equation

$$2x\,P(x-2) - P(x^2) = 3x^2 - 22x + q$$

holds for all real numbers x.

Category B (Team Competition)

B–T–1

Let us consider a triangle ABC with altitudes $h_a = 24$ and $h_b = 32$. Prove that for the third its altitude h_c the following inequalities

$$13 < h_c < 96$$

are fulfilled.

B–T–2

A square piece of paper $ABCD$ is folded such that the corner A comes to lie on the midpoint M of the side BC. The resulting crease intersects AB in X and CD in Y. Show that $|AX| = 5\,|DY|$.

B–T–3

Prove that the number $2010^{2011} - 2010$ is divisible by $2010^2 + 2011$.

Category C (Individual Competition)

C–I–1

Prove that the sum of the squares of eight consecutive odd positive integers is divisible by the number 8.

C–I–2

We are given a rectangle $ABCD$. Arbitrary triangles ABX and CDY are erected by its sides AB and CD. We define the mid-points P of AX, Q of BX, R of CY and S of DY. Prove that the line segments PR and QS have a common mid-point.

C–I–3

Let us consider a trapezoid $KLMN$ with sides of lengths $3, 3, 3, k$ with positive integer k. Determine the maximum area of such a trapezoid.

C–I–4

Prove that each positive integer $n \geq 6$ can be written as a sum of two positive integers, one of which is prime and the second of which is a composite number.

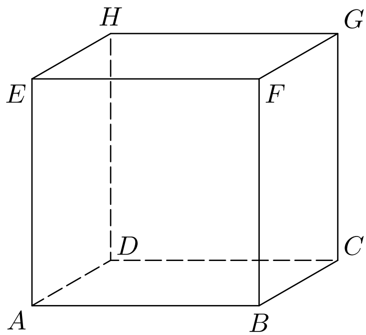
Category C (Team Competition)

C–T–1

Determine the number of all pairs (x, y) of decimal digits such that the positive integer in the form $\overline{xyx}$ is divisible by 3 and the positive integer in the form $\overline{yxy}$ is divisible by 4.

C–T–2

We are given a cube $ABCDEFGH$ (see picture) with edges of the length 5 cm. Determine the lengths of all altitudes of the triangle EDC.



C–T–3

The sum of all digits of a three-digit prime p_1 is a two-digit prime p_2. The sum of the digits of p_2 is a one-digit prime $p_3 > 2$. Find all triples (p_1, p_2, p_3) of such primes.

Solutions

Category A (Individual Competition)

A–I–1

Using Vièta's formulas we must determine the real numbers a, b, c such that the equations

$$a + b + c = -ab, \qquad ab + bc + ca = bc \qquad \text{and} \qquad abc = -ca$$

hold. After easy rewriting of these equations we have

$$a(b + 1) + (b + c) = 0, \qquad a(b + c) = 0 \qquad \text{and} \qquad ac(b + 1) = 0.$$

If $a = 0$, then $b + c = 0$ and we obtain the solution in the form $(a, b, c) = (0, b, -b)$ with an arbitrary real b.

If $a \neq 0$, then both $b + c = 0$ and $b + 1 = 0$ must be fulfilled. This implies $b = -1$ and $c = 1$. Thus we obtain the further solution $(a, b, c) = (a, -1, 1)$ with arbitrary real $a \neq 0$.

Conclusion. All solutions of the given problem are triples of mutually distinct real numbers (a, b, c) in the form $(0, -b, b)$ with $b \neq 0$ or in the form $(a, -1, 1)$ with $a \neq -1$, $a \neq 0$ and $a \neq 1$.

A–I–2

Let E be the point on the line AB such that $DB \parallel CE$. We can calculate the area P of the triangle AEC from Heron's formula in the following way

$$P = \sqrt{p(p - a)(p - b)(p - c)},$$

where a, b, c are lengths of sides of the triangle and

$$p = \frac{a + b + c}{2}.$$

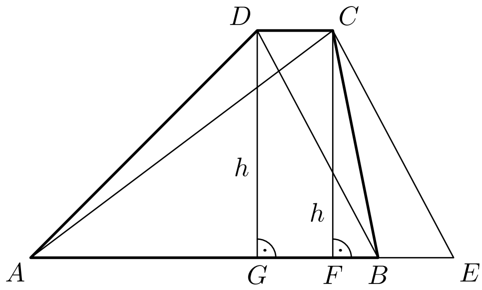
Then $a = |AE| = 28$, $b = |AC| = 25$, $c = |CE| = 17$. Further we have $p = \frac{1}{2}(28 + 25 + 17) = 35$ and $p - a = 7$, $p - b = 10$, $p - c = 18$.

$$P = \sqrt{35 \cdot 7 \cdot 10 \cdot 18} = \sqrt{5 \cdot 7 \cdot 7 \cdot 2 \cdot 5 \cdot 18} = 7 \cdot 5 \cdot 6 = 210.$$

We also have

$$P = \frac{1}{2} \cdot |AE| \cdot h = \frac{1}{2} \cdot 28 \cdot h = 14h,$$

where h is the altitude of the triangle AEC (i.e. the trapezoid $ABCD$). Thus $14h = 210$ holds, and therefore $h = 15$.



a) Let us consider the triangle ABC with the altitude $|CF| = 15$. From the Pythagorean theorem in the triangle AFC we obtain

$$|AF| = \sqrt{|AC|^2 - |CF|^2} = \sqrt{625 - 225} = \sqrt{400} = 20.$$

Then $|FB| = |AB| - |AF| = 23 - 20 = 3$. From the Pythagorean theorem in triangle GBD

$$|CB| = \sqrt{|CF|^2 + |FB|^2} = \sqrt{15^2 + 3^2} = \sqrt{225 + 9} = \sqrt{234}.$$

b) Let us now consider the triangle ABD with the altitude $|DG| = 15$. From the Pythagorean theorem in the triangle AFC we further obtain

$$|BG| = \sqrt{|BD|^2 - |DG|^2} = \sqrt{289 - 225} = \sqrt{64} = 8.$$

Then $|AG| = |AB| - |GB| = 23 - 8 = 15$. From the Pythagorean theorem in the triangle AGD we finally obtain

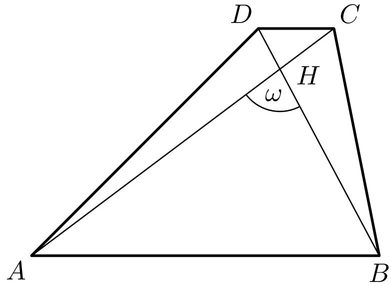
$$|AD| = \sqrt{|DG|^2 + |AG|^2} = \sqrt{15^2 + 15^2} = \sqrt{2 \cdot 15^2} = 15 \cdot \sqrt{2}.$$

Another solution. Let H denote the intersection point of the segments AC and BD and ω the angle AHB (see the figure). The lines AB and CD are parallel, and it follows that the triangles AHB and CHD are similar with coefficient $|AB| : |CD| = 23 : 5$. This yields

$$|AH| = 25 \cdot \frac{23}{28}, \quad |BH| = 17 \cdot \frac{23}{28}, \quad |CH| = 25 \cdot \frac{5}{28}, \quad \text{and} \quad |DH| = 17 \cdot \frac{5}{28}.$$

We can compute by the law of cosine in the triangle AHB

$$\cos\omega = \frac{|AH|^2 + |BH|^2 - |AB|^2}{2\,|AH|\cdot|BH|} = \frac{13}{85}.$$



For the supplementary angles to ω, we have

$$\cos\angle AHD = \cos\angle BHC = \cos(180^\circ - \omega) = -\cos\omega = -\frac{13}{85}.$$

Now we can use the law of cosines in the triangle BHC

$$|BC|^2 = |BH|^2 + |CH|^2 - 2\,|BH|\cdot|CH|\cos\angle BHC = 234 = 3^2\cdot 26$$

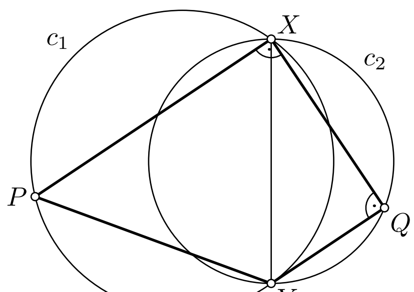
and in the triangle AHD

$$|AD|^2 = |AH|^2 + |DH|^2 - 2\,|AH|\cdot|DH|\cos\angle AHD = 450 = 15^2\cdot 2.$$

We therefore have $|BC| = 3\sqrt{26}$ and $|AD| = 15\sqrt{2}$.

A–I–3

As we can see in the figure, $\angle XQY = 90^\circ$ since XY is a diameter of c_2 and Q lies on c_2.



Considering the convex quadrilateral $PXQY$, we are given that PX and QY are parallel, making $PXQY$ a trapezoid. Since $\angle XQY = 90^\circ$, we therefore also have $\angle PXQ = 90^\circ$, and therefore $\angle XPY + \angle PYQ = 180^\circ$. Since the angle $\alpha = \angle XPY$ is independent of the choice of P on the greater arc XY of c_1, the angle $\angle PYQ = 180^\circ - \alpha$ is therefore also independent of the choice of P, as claimed.

A–I–4

Since $\sqrt{\sqrt{x}+2} \geq \sqrt{2}$ we get an estimate $y \geq 2 + \sqrt{2}$ and similarly $x \geq 2 + \sqrt{2}$. After squaring both given equations we obtain

$$\sqrt{x} + 2 = (y-2)^2,$$
$$\sqrt{y} + 2 = (x-2)^2,$$

After subtraction of the last two equations we obtain $\sqrt{x} - \sqrt{y} = (y-x)(x+y-4)$ which (after some manipulation) gives

$$(\sqrt{x} - \sqrt{y})\big(1 + (\sqrt{x} + \sqrt{y})(x+y-4)\big) = 0.$$

The second bracket on the left side of the previous equation is always nonzero and similarly (with regards to estimates in the beginning of the solution) in the case of the third bracket.

Therefore it follows $x = y = t^2$ with $t \geq \sqrt{2+\sqrt{2}}$, and thus $t > 1$. We obtain only one condition for t:

$$t + 2 = (t^2-2)^2, \quad \text{i.e.} \quad t^4 - 4t^2 - t + 2 = (t-2)(t^3 + 2t^2 - 1) = 0.$$

Since $t^3 + 2t^2 - 1 > 1 + 2 - 1 = 2$ holds for each $t > 1$, the last algebraic equation (of the 4th degree) has only one root $t > 1$, namely $t = 2$ and thus $x = y = 4$.

Conclusion. The given system of equation has only one solution $(x, y) = (4; 4)$ in real numbers.

Remark. The proof of the equality $x = y$ ($x \geq 2, y \geq 2$) can also be shown in the following way: If $x > y$, then

$$\sqrt{x} + 2 = (y-2)^2 < (x-2)^2 = \sqrt{y} + 2, \quad \text{i.e.} \quad x < y,$$

which is a contradiction with $x > y$ (and similarly in the case $x < y$). Thus $x = y$ necessarily holds.

Category A (Team Competition)

A–T–1

Multilpying $y^3+5x^2 = 1$ by x yields $xy^3+5x^3 = x$. Taking the difference of this and the first equation $x^4 + 5x^3 = y$ yields

$$x(x^3 - y^3) = y - x,$$

and dividing this equation by $x - y$ yields

$$x(x^2 + xy + y^2) = -1 \quad \text{which is equvalent to} \quad x^3 + x^2y + xy^2 = -1,$$

as claimed.

A–T–2

Let $|BE| = |DF| = x$, and thus $|CE| = |CF| = 1 - x$. Using conditions from the given problem we can see that the equation

$$1 + x + \sqrt{1 + x^2} = 2\sqrt{1 + x^2} + \sqrt{2(1 - x)^2}$$

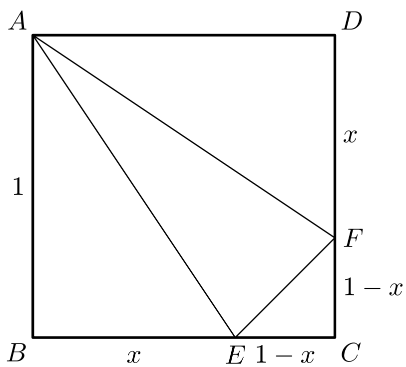
must be true. After double squaring of this equation and further rewriting we get the following equation:

$$x^4 + 2x^3 - 7x^2 + 2x + 1 = 0,$$

i.e.

$$0 = (x^4 + 2x^3 + 3x^2 + 2x + 1) - 10x^2 = (x^2 + x + 1)^2 - 10x^2.$$

After factorization of the left side of the last equation we see that there exists exactly one root x of this equation for $0 < x < 1$.



Conclusion. The solution of the given problem is

$$x = \frac{1}{2}\left(\sqrt{10} - 1 - \sqrt{7 - 2\sqrt{10}}\right).$$

A–T–3

It is easy to check that $n \geq 0$. From

$$\begin{aligned} 4^n &\equiv 1 \pmod 9 \quad \text{for} \quad n \equiv 0 \pmod 3, \\ 4^n &\equiv 4 \pmod 9 \quad \text{for} \quad n \equiv 1 \pmod 3, \\ 4^n &\equiv 7 \pmod 9 \quad \text{for} \quad n \equiv 2 \pmod 3, \end{aligned}$$

$1899 = 3^2 \cdot 211$ (211 is a prime) and

$$m^3 \equiv 0, 1, 8 \pmod 9 \quad \text{for} \quad m \equiv 0, 1, 2 \pmod 3$$

we see $n = 3k$ and $m = 3l + 1$, where k, l are integers, $k \geq 0$. We can rewrite the solved equation to the form

$$1899 = 4^{3k} - m^3 = \left(4^k - m\right)\left(4^{2k} + m \cdot 4^k + m^2\right).$$

The expression $4^{2k} + m \cdot 4^k + m^2 = (4^k + \frac{1}{2}m)^2 + \frac{3}{4}m^2$ takes only positive values, so $4^k - m > 0$. Moreover $4^k - m \equiv 1 - 1 \equiv 0 \pmod 3$ and $4^{2k} + m \cdot 4^k + m^2 \equiv 1 + 1 + 1 \equiv 0 \pmod 3$. Therefore we have to solve two systems of equations

$$\begin{aligned} 4^k - m &= 3, \\ 4^{2k} + m \cdot 4^k + m^2 &= 633 \end{aligned}$$

and

$$\begin{aligned} 4^k - m &= 633, \\ 4^{2k} + m \cdot 4^k + m^2 &= 3. \end{aligned}$$

It is easy to check that the second one has no solution. Substituting $4^k = 3 + m$ in the second equation of the first system we obtain a quadratic equation $3(m^2 + 3m + 3) = 633$ with roots -16, 13. The equation $4^k = 3 + m$ has an integer solution only for $m = 13$, and this solution is $k = 2$, so $n = 6$.

The proposed problem has the unique solution $m = 13$ and $n = 6$.

Category B (Individual Competition)

B–I–1

We write the number $\overline{abcabcabc}$ in the form

$$\overline{abcabcabc} = \overline{abc} \cdot 1001001.$$

Because 29 is not divisor of 1001001, it follows that $29 \mid \overline{abc}$.

The least three digit number divisible by 29 is $4 \cdot 29 = 116$, the greatest one is $34 \cdot 29 = 986$. So 31 numbers satisfy all premises.

B–I–2

Rewriting the left side of the given equation by the way $4^a - b^2 = (2^a)^2 - b^2$ we have

$$(2^a)^2 - b^2 = (2^a - b)(2^a + b) = 7.$$

Since $2^a + b$ is a positive integer and $2^a - b$ is an integer, it must be $2^a - b = 1$ and $2^a + b = 7$. From the first equation we have $b = 2^a - 1$. Substituting b in the given equation we get

$$4^a - (2^a - 1)^2 = 7, \qquad \text{i.e.} \qquad 2^{a+1} = 8.$$

This yields $a = 2$ and $b = 3$.

Conclusion. The unique (positive integer) solution of the given equation is the pair $(a, b) = (2; 3)$.

B–I–3

We will consider three possibilities.

a) The points A, B, C, D lie on the same line ℓ. Further we will consider two possibilities:
   - If $|AB| = |CD|$, then each point of the plane (excepting ℓ) solves our problem.
   - If $|AB| \neq |CD|$, then there exists no solution.

b) The line AB is parallel with CD (both lines have no common point). We then also have two possibilities.
   - If $|AB| = |CD|$, then V is each point of the mid-parallel a of the two parallel lines AB and CD.

- If $|AB| \neq |CD|$, then V is each of points of two parallel lines a_1 and a_2 with AB and CD such that $\mathrm{d}(AB, a_i) \cdot |AB| = \mathrm{d}(CD, a_i) \cdot |CD|$, $i = 1, 2$, (see Fig. 1).

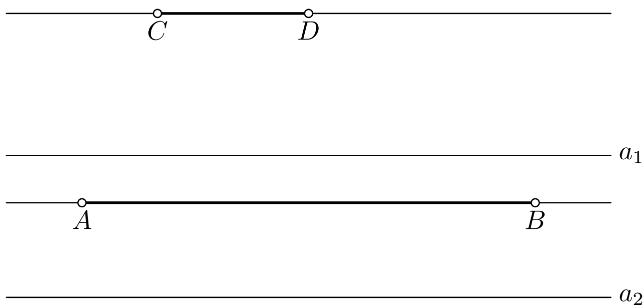


Fig. 1

c) The line AB intersects the line CD at the point P. Since the triangles ABV and CDV have the same area, it holds necessary

$$\mathrm{d}(AB, V) \cdot |AB| = \mathrm{d}(CD, V) \cdot |CD|,$$

i.e. $\mathrm{d}(CD, V) : \mathrm{d}(AB, V) = |AD| : |CD| = \text{const}$. It means all points V lie on a line. It is easy to see that V lies either on the line l_1 or on the line l_2 (without the point P). Nevertheless (for construction of l_1 and l_2), it holds $|AB| = |PB'| = |PB''|$ and $|CD| = |PD'|$ (see Fig. 2).

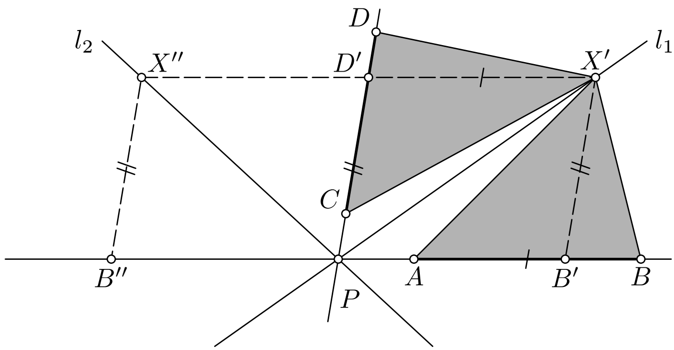


Fig. 2

B–I–4

We rewrite the given equation as

$$P(x^2) + 3x^2 - 22x + q = 2x\,P(x-2)$$

and consider possible degrees of the polynomial $P(x)$. If $\deg P(x) = 0$ holds, the left side of the equation is of degree 2 and the right side of degree 1, which is impossible. If $\deg P(x) = n \geq 2$ holds, the left side is of degree $2n$ and the right side of degree $n + 1$, which is not possible, since we certainly have $2n > n + 1$. It therefore follows that $\deg P(x) = 1$ must hold for any polynomial $P(x)$ fulfilling the requirements of the problem.

We can therefore write $P(x) = ax + b$, and substituting this expression yields

$$ax^2 + b + 3x^2 - 22x + q = 2x\,(a(x-2) + b),$$

which is equivalent to

$$(3-a)x^2 + (b+q) = (-4a + 2b + 22)x.$$

We therefore have $3 - a = 0 \Leftrightarrow a = 3$, $-4a + 2b + 22 = 0 \Leftrightarrow b = -5$ and $b + q = 0 \Leftrightarrow q = 5$. It follows that the only possible polynomial fulfilling the given conditions is $P(x) = 3x - 5$, and this is only a solution for $q = 5$.

Category B (Team Competition)

B–T–1

For the area S of a triangle ABC we have

$$S = \frac{1}{2} \cdot a \cdot h_a = \frac{1}{2} \cdot b \cdot h_b = \frac{1}{2} \cdot c \cdot h_c.$$

It follows

$$a = \frac{2S}{h_a} = \frac{2S}{24}, \quad b = \frac{2S}{h_b} = \frac{2S}{32} \quad \text{and} \quad c = \frac{2S}{h_c}.$$

From the triangle inequalities $|a - b| < c < a + b$ we obtain

$$\frac{2S}{24} - \frac{2S}{32} < \frac{2S}{h_c} < \frac{2S}{24} + \frac{2S}{32},$$

and dividing both sides by $2S$

$$\frac{1}{24} - \frac{1}{32} < \frac{1}{h_c} < \frac{1}{24} + \frac{1}{32},$$
$$\frac{1}{96} < \frac{1}{h_c} < \frac{7}{96}.$$

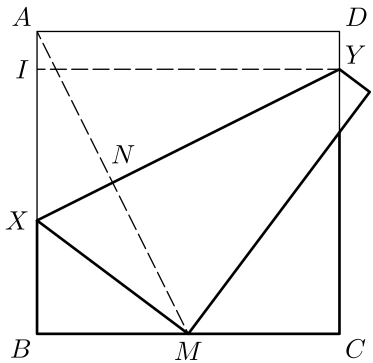
It follows

$$96 > h_c > \frac{96}{7} = 13 + \frac{5}{7} > 13,$$

which was to be shown.

B–T–2

As shown in the figure, let N be the point of intersection of AM and the crease line XY. Furthermore, let the lengths of the sides of $ABCD$ be equal to 1.



Since $|BM| = \frac{1}{2}$, we have $|AM| = \frac{1}{2}\sqrt{5}$, and therefore $|AN| = \frac{1}{2}|AM| = \frac{1}{4}\sqrt{5}$. The right-angled triangles ABM and AXN share the angle in A, and are therefore similar, and we have $|AX| : |AN| = |AM| : |AB| \Leftrightarrow |AX| : \left(\frac{1}{4}\sqrt{5}\right) = \frac{1}{2}\sqrt{5} : 1$, and therefore $|AX| = \frac{5}{8}$.

If I is an orthogonal projection of Y on AB, the triangles YIX and ABM are congruent, since their sides are pairwise orthogonal, and $|AB| = |YI| = 1$ holds. We therefore have $|IX| = |BM| = \frac{1}{2}$, and therefore $|DY| = |AX| - |IX| = \frac{5}{8} - \frac{1}{2} = \frac{1}{8}$, and we see that $|AX| = 5\,|DY|$ holds as claimed.

B–T–3

We first note that

$$x^{2010} - 1 = (x^3 - 1)(x^{2007} + x^{2004} + \ldots + 1) \quad \text{and} \quad x^3 - 1 = (x - 1)(x^2 + x + 1)$$

both hold. We therefore have

$$(x^2 + x + 1) \mid (x^3 - 1) \quad \text{and} \quad (x^3 - 1) \mid (x^{2010} - 1),$$

and therefore also

$$(x^2 + x + 1) \mid (x^{2010} - 1).$$

Setting $x = 2010$ therefore yields

$$(2010^2 + 2010 + 1) \mid (2010^{2010} - 1),$$

and multiplying by 2010 therefore yields

$$(2010^2 + 2011) \mid (2010^{2011} - 2010)$$

as claimed.

Category C (Individual Competition)

C–I–1

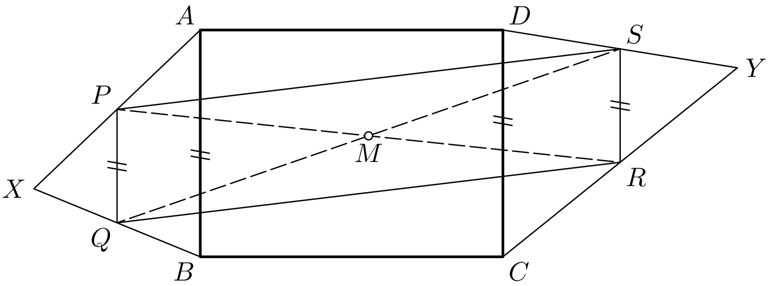
We can write any eight consecutive odd positive integers in the form: $2k-7$, $2k-5$, $2k-3$, $2k-1$, $2k+1$, $2k+3$, $2k+5$, $2k+7$ with a positive integer k ($k \geq 4$). The sum of the squares of these eight integers is therefore

$$\begin{aligned}(2k-7)^2 + (2k-5)^2 + (2k-3)^2 + (2k-1)^2 &+ \\ + (2k+1)^2 + (2k+3)^2 + (2k+5)^2 + (2k+7)^2 &= \\ = 32k^2 + 168 = 8(4k^2 + 21)&,\end{aligned}$$

which proves the claim.

C–I–2

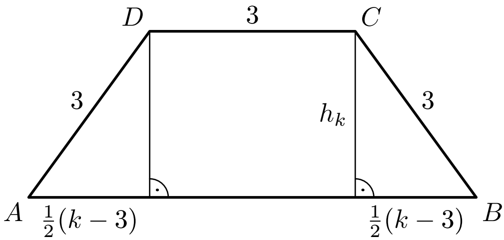
The figure shows the situation as described in the problem.



Since P and Q are the midpoints of AX and BX respectively, the line segment PQ results from the line segment AB by homothety with center X and factor $\frac{1}{2}$. It follows that the line segment PQ is parallel to AB and half as long as AB. By the same reasoning on the other side of the rectangle, the line segment RS is parallel to CD and half as long. Since AB and CD are opposite sides of a rectangle, they are parallel and equally long. This therefore also holds for PQ and RS. We see that $PQRS$ must be a parallelogram, and it therefore follows that the diagonals PR and QS of this parallelogram have a common midpoint.

C–I–3

For such a trapezoid to exist, the inequalities $1 \le k \le 8$ must be fulfilled. Let P_k denote the area of the trapezoid for side length k. We wrote that the length of an altitude of trapezoids for $k = 2$ and $k = 4$ are the same. Then $P_2 < P_4$. Similarly the length of an altitude of trapezoids for $k = 1$ and $k = 5$ are the same. Then $P_1 < P_5$.



Now, we will compute areas P_k of the trapezoids for $k = 3, 4, 5, 6, 7, 8$.

For $3 \le k \le 8$ we have $|AE| = |FB| = \frac{1}{2}(k-3)$. Then for the lengths of their altitudes h_k we have

$$h_k = \sqrt{BD^2 - FB^2} = \sqrt{9 - \frac{(k-3)^2}{4}}$$

and the area

$$P_k = \frac{k+3}{2} \cdot \sqrt{9 - \frac{(k-3)^2}{4}}.$$

For $k = 3$ the trapezoid is the square and

$$P_3 = \sqrt{81} = \frac{\sqrt{1296}}{4}.$$

For $k = 4$ we have

$$P_4 = \frac{\sqrt{1715}}{4}.$$

For $k = 5$ we have

$$P_5 = \sqrt{128} = \frac{\sqrt{2048}}{4}.$$

For $k = 6$ we have

$$P_6 = \frac{\sqrt{2187}}{4}.$$

For $k = 7$ we have

$$P_7 = \sqrt{125} = \frac{\sqrt{2000}}{4}.$$

For $k = 8$ we have

$$P_8 = \frac{\sqrt{1331}}{4}.$$

The maximal area of considered trapezoids is therefore for $k = 6$.

C–I–4

If n is an even positive integer, we can write such $n \geq 6$ as the sum $2 + (n - 2)$, in which the first summand is the prime and the second one is an even positive integer ($n - 2 \geq 4$), i.e. $n - 2$ is a composite number.

Similarly, for each odd positive integer $n \geq 7$ we can write n as the sum $3 + (n - 3)$, in which the first summand is the prime and second one ($n - 3 \geq 4$) is an even, so a composite, number.

This concludes the proof.

Category C (Team Competition)

C–T–1

We can use criteria of divisibility by positive integers 3 and 4. Each positive integer in the form $\overline{yxy}$ is divisible by 4 if and only if the number $\overline{xy}$ is divisible by 4 and simultaneously $y \neq 0$. Hence

$$\overline{xy} \in \{12, 16, 24, 28, 32, 36, 44, 48, 52, 56, 64, 68, 72, 76, 84, 88, 92, 96\}.$$

A positive integer in the form $\overline{xyx}$ is divisible by 3 if and only if the sum of its digits is divisible by 3, i.e. the sum $2x + y$ must be divisible by 3. After checking all possible pairs of positive integers we obtain only six possibilities:

$$\overline{xy} \in \{28, 36, 44, 52, 88, 96\}.$$

Conclusion. Solutions of the given problem are the following pairs if positive integers $(2;8)$, $(3;6)$, $(4;4)$, $(5,2)$, $(8;8)$, $(9;6)$, i.e. we have in total 6 solutions.

C–T–2

Since $|CD| = 1$, $|DE| = \sqrt{2}$ and $|EC| = \sqrt{3}$, the triangle EDC is right-angled (by the reverse Pythagorean theorem). Therefore two of its altitudes are congruent with legs, i.e. 1 and $\sqrt{2}$. We can calculate the last altitude d (from vertex D to the hypotenuse EC using double counting of the area S of the right-angled triangle EDC. We have

$$S = \frac{1}{2} \cdot 1 \cdot \sqrt{2} = \frac{1}{2} \cdot \sqrt{3} \cdot d, \quad \text{and thus} \quad d = \frac{\sqrt{6}}{3}.$$

Conclusion. The triangle EDC has altitudes of lengths 1, $\sqrt{2}$ and $\frac{1}{3}\sqrt{6}$.

C–T–3

The sum of the digits of each three-digit number is at most $3 \cdot 9 = 27$. Among all two-digit positive integers which are not greater than 27 there exist only five two-digit primes $(11, 13, 17, 19, 23)$. The sums of the digits of these five two-digit primes are $2, 4, 8, 10, 5$, respectively. Among them there exists exactly one, which is a prime greater than 2. We have therefore $p_3 = 5$ and $p_2 = 23$.

After checking of all decompositions of the number $p_2 = 23$ into three summands which are decimal digits we obtain four solutions for which p_1 is also a prime. There are the three-digit numbers $599, 797, 887$ and 977.

Conclusion. There exist four triples (p_1, p_2, p_3) of primes satisfying the conditions of the given problem:

$$(p_1, p_2, p_3) \in \{(559, 23, 5); (797, 23, 5); (887, 23, 5); (977, 23, 5)\}.$$

Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Simona Domesová	GMK Bílovec	5	8	8	7	**28**
2.	Bianca-Ioana Voicu	CNC Ploieşti	3	8	8	7	**26**
3.	Wojciech Lis	I LO Chorzów	8	8	8	0	**24**
4.	Martin Broušek	GJŠ Přerov	6	8	8	1	**23**
	Martin Unger	BRG Kepler Graz	0	8	8	7	**23**
6.	Andreea-Elena Cârciumaru	CNC Ploieşti	4	8	8	2	**22**
7.	Josef Malík	GMK Bílovec	0	8	8	5	**21**
8.	Michał Chyra	I LO Chorzów	6	0	8	6	**20**
9.	Petr Boroš	GMK Bílovec	0	3	8	8	**19**
10.	Valentin Borzan	BRG Kepler Graz	0	8	8	1	**17**
11.	Bartosz Badura	I LO Chorzów	8	0	8	0	**16**
	Jakub Solovský	GMK Bílovec	4	0	8	4	**16**
13.	Karel Kraus	GJŠ Přerov	0	0	8	6	**14**
14.	Andrei-Răzvan Mareşu	CNC Ploieşti	0	8	1	4	**13**
15.	Krzysztof Paprotny	I LO Chorzów	0	8	1	2	**11**
	Golo Wimmer	BRG Kepler Graz	0	2	8	1	**11**
17.	Stephan Meighen-Berger	BRG Kepler Graz	1	7	0	2	**10**
	Vojtěch Miloš	GJŠ Přerov	0	0	6	4	**10**
19.	Lukáš Langer	GJŠ Přerov	0	0	8	1	**9**
20.	Miriam-Elena Şerban	CNC Ploieşti	1	1	2	0	**4**

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Tomasz Cieśla	III LO Chorzów	8	8	8	8	**32**
	Radu-Ştefan Voroneanu	CNC Ploieşti	8	8	8	8	**32**
3.	Bernd Prach	BRG Kepler Graz	8	6	4	6	**24**
4.	V.-P. Veigang-Rădulescu	CNC Ploieşti	8	8	1	1	**18**
5.	Theodor-Daniel Nedelcu	CNC Ploieşti	8	8	1	0	**17**
6.	Pavel Trutman	GMK Bílovec	5	8	1	0	**14**
7.	Constantin Albu	CNC Ploieşti	4	8	1	0	**13**
8.	Clemens Andritsch	BRG Kepler Graz	4	7	1	0	**12**
9.	Manuel Gruber	BRG Kepler Graz	8	2	0	0	**10**
10.	Barbora Mólová	GMK Bílovec	0	8	0	1	**9**
11.	Šimon Rozsíval	GJŠ Přerov	0	7	1	0	**8**
	Maciej Wojsyk	I LO Chorzów	0	0	2	6	**8**
13.	Tomasz Depta	I LO Chorzów	4	2	1	0	**7**
	Eva Gocníková	GJŠ Přerov	0	5	0	2	**7**
15.	Florian Krach	BRG Kepler Graz	3	1	2	0	**6**
16.	Elizabeth Brázdilová	GMK Bílovec	0	1	3	0	**4**
	Kateřina Solovská	GMK Bílovec	2	2	0	0	**4**
18.	Alena Harlenderová	GJŠ Přerov	0	0	2	0	**2**
	Anna Kula	I LO Chorzów	1	1	0	0	**2**
	Klára Švarcová	GJŠ Přerov	1	1	0	0	**2**

Category C (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Kinga Turlej	I LO Chorzów	8	8	6	8	**30**
2.	Andrei Matei	CNC Ploieşti	8	8	5	8	**29**
3.	Diana-Maria Cremarenco	CNC Ploieşti	8	7	8	5	**28**
4.	Łukasz Ławniczak	I LO Chorzów	8	8	2	8	**26**
5.	Heinz Prach	BRG Kepler Graz	8	8	0	8	**24**
	Roxana-Mihaela Săvulescu	CNC Ploieşti	8	8	0	8	**24**
7.	Magdalena Marcinowicz	I LO Chorzów	2	8	6	1	**17**
8.	Jarosław Socha	I LO Chorzów	8	0	0	8	**16**
9.	Pavel Berger	GMK Bílovec	2	6	0	7	**15**
10.	Jan Krejčí	GMK Bílovec	8	0	0	6	**14**
	Bogdan-Constantin Ioniţă	CNC Ploieşti	6	8	0	0	**14**
12.	Martin Sládeček	GJŠ Přerov	1	7	0	0	**8**
	Michaela Jandeková	GMK Bílovec	2	6	0	0	**8**
	Martin Rychtárik	GMK Bílovec	0	0	0	8	**8**
15.	Benjamin von Berg	BRG Kepler Graz	0	7	0	0	**7**
16.	Matěj Tomešek	GJŠ Přerov	1	1	4	0	**6**
17.	Zuzana Gocníková	GJŠ Přerov	1	0	0	3	**4**
18.	Felix Feistritzer	BRG Kepler Graz	1	0	0	0	**1**
	Sarah Fruhmann	BRG Kepler Graz	0	1	0	0	**1**
	Tomáš Kremel	GJŠ Přerov	1	0	0	0	**1**

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	CNC Ploieşti	8	8	1	**17**
2.	I LO Chorzów	8	8	0	**16**
3.	GMK Bílovec	8	7	0	**15**
4.	BRG Kepler Graz	0	8	1	**9**
	GJŠ Přerov	0	8	1	**9**

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	CNC Ploieşti	8	8	8	**24**
2.	I LO Chorzów	7	8	8	**23**
3.	GJŠ Přerov	8	8	0	**16**
4.	BRG Kepler Graz	7	5	0	**12**
5.	GMK Bílovec	0	1	0	**1**

Category C (Team Competition)

Rank	School	1	2	3	∑
1.	GMK Bílovec	8	8	8	**24**
	I LO Chorzów	8	8	8	**24**
3.	CNC Ploieşti	8	8	7	**23**
4.	BRG Kepler Graz	8	3	7	**18**
5.	GJŠ Přerov	2	7	1	**10**