Palacký University Olomouc, Faculty of Science
Jakub Škoda Gymnasium Přerov

MATHEMATICAL DUEL ’11

Jaroslav Švrček
Pavel Calábek
Robert Geretschläger
Józef Kalinowski
Jacek Uryga

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Preface

The 19th International Mathematical Duel was held from 22–25 March 2011 in Přerov. In this year the competition was organized by Jakub Škoda Gymnasium Přerov in cooperation with Faculty of Science of Palacký University Olomouc.

Five school-teams from Austria, Czech Republic, Poland and Italy took part in this traditional mathematical competition, namely from Bundesrealgymnasium Kepler, Graz, Gymnázium M. Koperníka, Bílovec, I Liceum Ogólnokształcące im. J. Słowackiego, Chorzów, Gymnázium J. Škody, Přerov and for the first time one team from Liceo Scientifico Statale A. Labriola, Roma-Ostia (Italy) as guests.

As usual the competition was provided in the three categories (A – contestants of the last two years, B – contestants of the 5th and 6th years, and C – contestants of the 3rd and 4th years of eight-year grammar school). Twelve contestants (more precisely 4 in any category) of any school took part in this competition, i.e. 60 contestants in total.

This booklet contains all problems with solutions and results of the 19th International Mathematical Duel from the year 2011.

Authors

Problems

Category A (Individual Competition)

A–I–1

Solve in the domain of real numbers the following system of equations

$$x^4 + 1 = 2yz,$$
$$y^4 + 1 = 2zx,$$
$$z^4 + 1 = 2xy.$$

Jaroslav Švrček

A–I–2

We are given a trapezoid $ABCD$ with $AB \parallel CD$ and $|AB| = 2\,|CD|$. Let M be the common point of the diagonals AC and BD and E the midpoint of AD. Lines EM and CD intersect in P. Prove that $|CP| = |CD|$ holds.

Robert Geretschläger

A–I–3

Let a, b, p, q and $p\sqrt{a} + q\sqrt{b}$ be positive rational numbers. Prove that numbers $\sqrt{a}$ and $\sqrt{b}$ are also rational.

Jacek Uryga

A–I–4

We are given an acute-angled triangle ABC. Let us consider a triangle KLM with vertices in the feet of the altitudes of the given triangle. Prove that the orthocenter of triangle ABC is equal to the incenter of triangle KLM.

Józef Kalinowski

Category A (Team Competition)

A–T–1

Determine all polynomials $f(x) = x^2 + px + q$ with integer coefficients p, q such that $f(x)$ is a perfect square for infinitely many integers x.

Jacek Uryga

A–T–2

Let c be a circle with center O and radius r and ℓ a line containing O. Further let P and Q be points on c symmetric with respect to ℓ. X is a point on c such that $OX \perp \ell$ and A, B are points of intersection of XP with ℓ, XQ with ℓ respectively. Prove that $|OA| \cdot |OB| = r^2$ holds.

Robert Geretschläger

A–T–3

Peter throws two dice together and then always writes the number of all showing dots on the blackboard. Find the least number k with the following property: After k throws Peter can always choose some of the written numbers, such that their product has remainder 1 after division by 13.

Pavel Calábek

Category B (Individual Competition)

B–I–1

Let A be a six-digit positive integer which is formed by using two digits x, y only. Further let B be a six-digit positive integer resulting from A if all digits x are replaced by y and simultaneously all digits y are replaced by x. Prove that the sum $A + B$ is divisible by 91.

Józef Kalinowski

B–I–2

An isosceles right-angled triangle EBC with right angle at C and $|BC| = 2$ is given in the plane. Determine all possible areas of trapezoids $ABCD$ ($AB \parallel CD$) in which E is the mid-point of AD.

Jaroslav Švrček

B–I–3

Prove that there exist infinitely many solutions of the equation

$$2^x + 2^{x+3} = y^2$$

in the domain of positive integers.

Jaroslav Švrček

B–I–4

We are given a common external tangent line t to circles $c_1(O_1; r_1)$ and $c_2(O_2; r_2)$ which have no common point and lie in the same half-plane defined by t. Let d be the distance between the tangent points of circles c_1 and c_2 with the line t. Determine the smallest possible length of a broken line AXB (i.e. the union of line segments AX and XB), such that A belongs to c_1, B belongs to c_2 and X lies on t.

Jaroslav Švrček

Category B (Team Competition)

B–T–1

Solve in the domain of positive integers the following equation

$$\frac{2}{x^2} + \frac{3}{xy} + \frac{4}{y^2} = 1.$$

Józef Kalinowski

B–T–2

Let E be the mid-point of the side CD of the convex quadrilateral $ABCD$ in the plane. Prove the following statement: If the area of the triangle AEB is half of the area of $ABCD$, then $ABCD$ is a trapezoid.

Jacek Uryga

B–T–3

Determine all real solutions of the following system of equations

$$\begin{aligned} 2a - 2b &= 29 + 4ab, \\ 2c - 2b &= 11 + 4bc, \\ 2c + 2a &= 9 - 4ca. \end{aligned}$$

Robert Geretschläger

Category C (Individual Competition)

C–I–1

The number n has the following properties:

a) the product of all its digits is odd,
b) the sum of the squares of its digits is even.

Prove that the number of digits in n cannot be equal to 2011.

Robert Geretschläger

C–I–2

Let ABC be a right-angled triangle with hypotenuse AB. Determine measures of its angles at A and B if the angle bisector at B divides the opposite side AC at a point D such that $|AD| : |CD| = 2 : 1$.

Jaroslav Švrček

C–I–3

Determine the number of all ten-digit numbers which are divisible by 4 and which are written using only the digits 1 and 2.

Józef Kalinowski

C–I–4

Let p, q be two parallel lines in the plane and A a point lying outside of the strip bounded by the lines p and q. Construct a square $ABCD$ such that its vertices B, D lie on p and q, respectively.

Pavel Calábek

Category C (Team Competition)

C–T–1

Determine all pairs (x, y) of positive integers satisfying the following equation
$$(x+y)^2 = 109 + xy.$$

Józef Kalinowski

C–T–2

An isosceles triangle ABC with the base $|AB| = \sqrt{128}$ is given in the plane. The foot of its altitude from A divides the side BC into two parts in the ratio $1:3$ of their lengths. Determine the perimeter and area of this triangle.

Józef Kalinowski

C–T–3

Find all positive integers n such that the number $n^3 - n$ is divisible by 48.

Jaroslav Švrček

Solutions

Category A (Individual Competition)

A–I–1

For any real a, the inequality $2a^2 \le a^4 + 1$ holds. We therefore obtain the following estimates for left-hand sides of all equations of the given system:

$$2x^2 \le x^4 + 1 = 2yz,$$
$$2y^2 \le y^4 + 1 = 2zx,$$
$$2z^2 \le z^4 + 1 = 2xy.$$

Adding up all three inequalities (and dividing by 2) we have

$$x^2 + y^2 + z^2 \le xy + yz + zx. \tag{1}$$

On the other hand we know, that the inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx \tag{2}$$

is true. This follows immediately from the evident inequality

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \ge 0. \tag{3}$$

Therefore from (1) and (2) we have $x^2+y^2+z^2 = xy+yz+zx$. From the inequality (3) we can see that equality holds if and only if $x = y = z$. Thus we will solve the following biquadratic equation $x^4 + 1 = 2x^2$. It is easy to see that this equation has only two real roots, namely 1 and −1.

Conclusion. After checking (which is a part of this solution) we can see that the given system of equations has only two real solutions: $(x, y, z) = (1, 1, 1)$ and $(x, y, z) = (-1, -1, -1)$.

Remark. We can use another way to prove $x = y = z$. Multiplying the subsequent equations by x, y and z, respectively we obtain

$$x^5 + x = 2xyz,$$
$$y^5 + y = 2xyz,$$
$$z^5 + z = 2xyz.$$

Therefore $x^5 + x = y^5 + y = z^5 + z$. Since the function $f(t) = t^5 + t$ is increasing in the whole domain (as a sum of two increasing functions), the previous equality holds if and only if $x = y = z$.

A–I–2

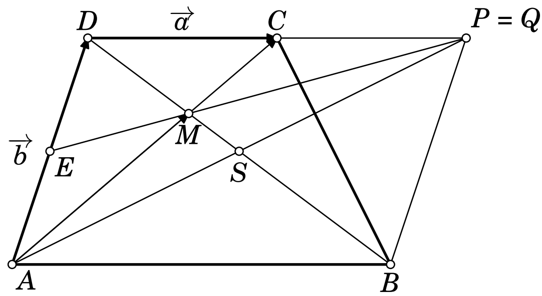
Let S be the mid-point of the diagonal BD. Further let Q be a point of intersection of the line AS with CD. Q is therefore a vertex of a paralelogram $ABQD$ (see picture). Since $|AB| = 2\,|CD|$, the mid-point of the side DQ is C. From the similarity of the triangles ABM and CDM we further obtain

$$|BM| : |DM| = |AB| : |CD| = 2 : 1$$

and also

$$|BM| : |DM| = |DM| : |MS| = 2 : 1.$$

Thus, the point M of intersection of diagonals AC and BD of the given trapezoid $ABCD$ must simultaneously be a centroid of the triangle AQD. Its median EQ is therefore collinear with EM. Thus $Q = P$, and the proof is finished.



Another solution. Let $\overrightarrow{DC} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$. We then have $\overrightarrow{AE} = \frac{1}{2}\vec{b}$ and $\overrightarrow{AM} = \frac{2}{3}(\vec{a} + \vec{b})$, since triangles MAB and MCD are similar with ratio 2 : 1. We therefore have $\overrightarrow{EM} = \frac{2}{3}\vec{a} + \frac{1}{6}\vec{b}$.

The vector $\overrightarrow{DP}$ can now be written in two ways, and we have

$$-\frac{1}{2}\vec{b} + \lambda\left(\frac{2}{3}\vec{a} + \frac{1}{6}\vec{b}\right) = \mu\,\vec{a},$$

and comparing coefficients therefore yields $\lambda = 3$, and thus $\mu = 2$. We see that DP is twice as long as DC, as claimed.

A–I–3

Let us observe that $p\sqrt{a}+q\sqrt{b}>0$ and

$$p^2a-q^2b=(p\sqrt{a}+q\sqrt{b})(p\sqrt{a}-q\sqrt{b}).$$

Hence

$$p\sqrt{a}-q\sqrt{b}=\frac{p^2a-q^2b}{p\sqrt{a}+q\sqrt{b}}$$

.

Since both the numerator and the denominator of the fraction are rational, so is the number $p\sqrt{a}-q\sqrt{b}$.

The rationality of $\sqrt{a}$ and $\sqrt{b}$ results now from the following two equalities:

$$\sqrt{a}=\frac{(p\sqrt{a}+q\sqrt{b})+(p\sqrt{a}-q\sqrt{b})}{2p},$$

$$\sqrt{b}=\frac{(p\sqrt{a}+q\sqrt{b})-(p\sqrt{a}-q\sqrt{b})}{2q}$$

and the rationality of the numbers p, q, $p\sqrt{a}+q\sqrt{b}$ and $p\sqrt{a}-q\sqrt{b}$.

A–I–4

Let D, E, F be the feet of the altitudes from vertices A, B, C of the given acute-angled triangle ABC and V be its orthocenter. First of all, we can see that

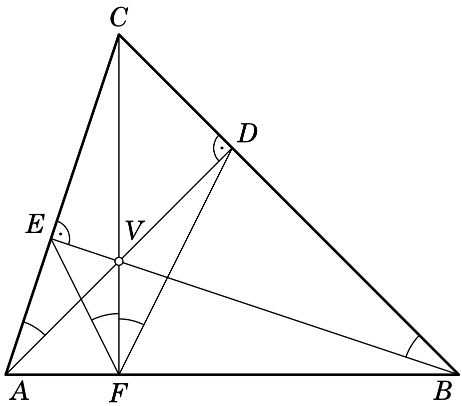
$$|\angle CAD|=|\angle CBE|=90^\circ-|\angle BCA|.$$

Since $VF\perp AB$ the quadrilaterals $AFVE$ and $BFVD$ are cyclic and therefore

$$|\angle EFV|=|\angle EAV|=|\angle CAD|=|\angle CBE|=|\angle DBV|=|\angle DFV|.$$

Cyclically we can also prove that

$$|\angle FDV|=|\angle EDV| \qquad \text{and} \qquad |\angle DEV|=|\angle FEV|,$$



which completes the proof.

Category A (Team Competition)

A–T–1

Let $f(x)$ be a perfect square for an integer x. Let us denote m such an integer that $x^2 + px + q = m^2$. We can rewrite this equation in the following way

$$(2x - 2m + p)(2x + 2m + p) = p^2 - 4q.$$

If $p^2 - 4q \neq 0$ then there exist only a finite number of integer factorizations of $p^2 - 4q$, so there exist only a finite number of the integer solutions x and m of the equation above.

On the other hand, if $p^2 - 4q = 0$, then p is even. For $x = m - \frac{1}{2}p$ (m is an arbitrary integer) follows

$$f(x) = (m - \tfrac{1}{2}p)^2 + p\,(m - \tfrac{1}{2}p) + q = m^2 - \tfrac{1}{4}(p^2 - 4q) = m^2,$$

so $f(x)$ is a perfect square.

Conclusion. $f(x)$ is the perfect square for infinitely many integers x if and only if p is even and $q = \frac{1}{4}p^2$.

Another solution. If the polynomial $f(x)$ satisfies the assumption, then the polynomial

$$g(x) = 4f(x) = 4x^2 + 4px + 4q = (2x+p)^2 + 4q - p^2$$

is also a perfect square for the same set of x as the polynomial $f(x)$.

For further investigations we need the proposition that for every integer $a > 0$ the interval $(a^2 - a, a^2 + a)$ contains exactly one perfect square, namely a^2.

To prove the proposition we note the following: for every non-negative $b \neq a$ we have $b \geq a + 1$ or $0 \leq b \leq a - 1$.

If $b \geq a+1$, then $b^2 \geq a^2 + 2a + 1 > a^2 + a$ and if $0 \leq b \leq a - 1$, then $b^2 \leq a^2 - 2a + 1 < a^2 - a$.

In both cases we see that $b^2 \notin (a^2 - a, a^2 + a)$, which proves the proposition.

Now, choose x such that $|2x + p| > |4q - p^2|$ and

$$g(x) = (2x+p)^2 + 4q - p^2$$

is a perfect square (we can do this, because there are infinitely many x for which $g(x)$ is a perfect square).

It is easy to see that the interval

$$(|2x+p|^2 - |2x+p|, |2x+p|^2 + |2x+p|) = ((2x+p)^2 - |2x+p|, (2x+p)^2 + |2x+p|)$$

contains the square $(2x+p)^2 + 4q - p^2$. Thus by the proposition

$$(2x+p)^2 + 4q - p^2 \quad \text{is equal to} \quad (2x+p)^2 \quad \text{and} \quad 4q - p^2 = 0.$$

Consequently we have

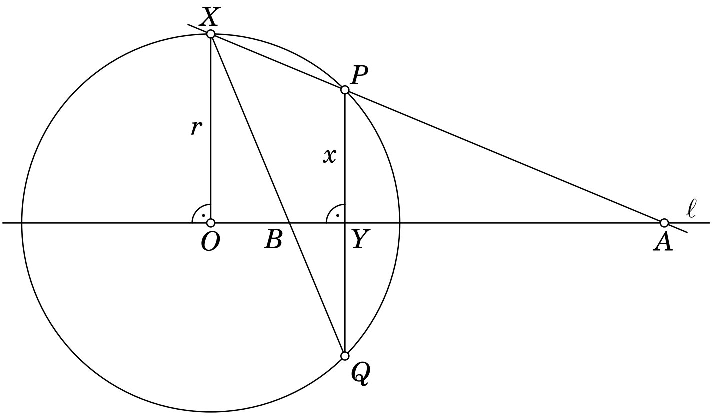
$$q = \tfrac{1}{4}p^2 \qquad \text{and} \qquad f(x) = x^2 + px + \tfrac{1}{4}p^2 = (x + \tfrac{1}{2}p)^2.$$

This polynomial is a perfect square for infinitely many integers if and only if the coefficient p is even.

A–T–2

We name $Y = PQ \cap \ell$ and $|PY| = |QY| = x$. Since $PQ \perp \ell$, the triangles AXO and APY are similar and we have

$$\frac{|OA|}{|OA|-|OY|} = \frac{r}{x} \quad \Rightarrow \quad x \cdot |OA| = r \cdot |OA| - r \cdot |OY| \quad \Rightarrow \quad |OA| = \frac{r \cdot |OY|}{r-x}.$$



Similarly, since the triangles BXO and BQY are similar, we have

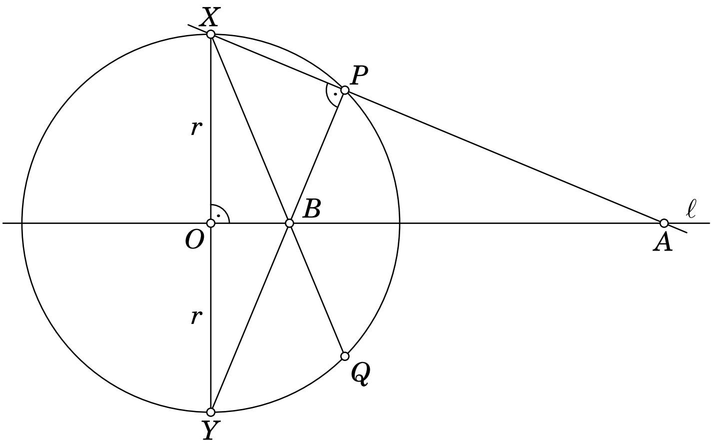
$$\frac{|OB|}{|OY|-|OB|} = \frac{r}{x} \quad \Rightarrow \quad x \cdot |OB| = r \cdot |OY| - r \cdot |OB| \quad \Rightarrow \quad |OB| = \frac{r \cdot |OY|}{r+x}.$$

It therefore follows that

$$|OA| \cdot |OB| = \frac{r^2 \cdot |OY|^2}{(r-x)(r+x)} = \frac{r^2(r^2-x^2)}{r^2-x^2} = r^2,$$

as claimed.

Another solution. Let Y be the reflection of X with respect to the line AO. Then the segment XY is diameter of the circle and so the angle XYP is right. The points X, Q are symmetric to Y, P with



respect to ℓ, so the point B lies on the segment YP. Now, observe that the triangles XAO, XYP and YBO are right-angled and the pairs of triangles XAO and XYP, XYP and YBO have a common acute angle. Thus we conclude that the all the mentioned triangles are similar.

By this similarity we have in particular

$$\frac{|AO|}{|XO|} = \frac{|YO|}{|BO|},$$

which proves the the required statement.

A–T–3

After each throw Peter writes on the blackboard some number from the set $\{2, 3, 4, \dots, 12\}$. If after every throw he writes a number 2 then the remainders of the product of the all 2's on the blackboard after division of 13 in n throws are in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12
$2^n \pmod{13}$	2	4	8	3	6	12	11	9	5	10	7	1

In this case Peter needs at least 12 throws. We will show that 12 throws are sufficient.

Let a_i be the number which Peter writes on the blackboard after the i-th throw. Let us denote $s_1 = a_1$, $s_2 = a_1a_2$, $s_3 = a_1a_2a_3$, $\dots$, $s_{12} = a_1a_2\dots a_{12}$. Since none of the numbers a_i is divisible by 13 then the remainders of s_i after division by 13 are from the set

$\{1, 2, 3, \ldots, 12\}$. If there exists an index i such that the remainder of s_i after division by 13 is 1, then the proof is finished. On the other hand 12 numbers $s_1, s_2, \ldots, s_{12}$ have remainders (after division by 13) in the set $\{2, 3, 4, \ldots, 12\}$ having 11 elements. Using the Pigeonhole Principle there exist two indices i, j $(1 \le i < j \le 12)$ such the numbers s_i and s_j have the same remainder after division by 13. In that case the difference $s_j - s_i$ is divisible by 13. But

$$s_j - s_i = a_1 a_2 \ldots a_j - a_1 a_2 \ldots a_i = a_1 a_2 \ldots a_i (a_{i+1} \ldots a_j - 1) = s_i (a_{i+1} \ldots a_j - 1).$$

Since s_i isn't divisible by 13 then $a_{i+1} \ldots a_j$ has remainder 1 after division by 13. It is easy to see that the product $a_{i+1} \ldots a_j$ has at least two factors since none of a_i has remainder 1.

Conclusion. The least possible number k of such throws is 12.

Category B (Individual Competition)

B–I–1

Let $A = \overline{c_5 c_4 c_3 c_2 c_1 c_0}$ and $B = \overline{d_5 d_4 d_3 d_2 d_1 d_0}$, where $c_i, d_i \in \{x, y\}$, $c_i \ne d_i$ for $i = 0, 1, 2, 3, 4, 5$ and $x, y \in \{1, \ldots, 9\}$ be non-zero distinct decimal digits.

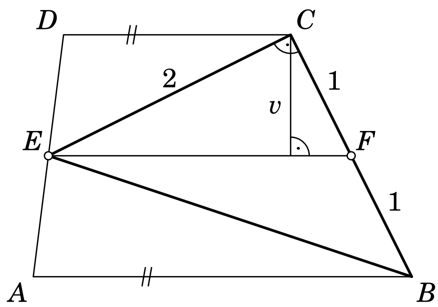
Since $c_i + d_i = x + y \ne 0$ for $i = 0, 1, 2, 3, 4, 5$ we can count the sum

$$\begin{aligned} A + B &= c_5 \cdot 10^5 + c_4 \cdot 10^4 + c_3 \cdot 10^3 + c_2 \cdot 10^2 + c_1 \cdot 10^1 + c_0 \cdot 10^0 \\ &\quad + d_5 \cdot 10^5 + d_4 \cdot 10^4 + d_3 \cdot 10^3 + d_2 \cdot 10^2 + d_1 \cdot 10^1 + d_0 \cdot 10^0 \\ &= (x + y) \cdot (10^5 + 10^4 + 10^3 + 10^2 + 10 + 1) = (x + y) \cdot 111111 \\ &= (x + y) \cdot 91 \cdot 1221. \end{aligned}$$

Thus the number $A + B$ is divisible by 91.

B–I–2

Let F be a mid-point of the side BC of the trapezoid $ABCD$. Let us consider the right-angled triangle EFC. Using Pythagoras theorem



we obtain the length of its hypotenuse EF. We obtain $|EF| = \sqrt{5}$. Using double counting of the twice the area of this triangle we obtain

$$\sqrt{5} \cdot v = |EF| \cdot v = |EC| \cdot |FC| = 2 \cdot 1 = 2.$$

From this equation it follows

$$v = \frac{2\sqrt{5}}{5},$$

where v is the altitude from the vertex C in the triangle EFC (see the picture). The area P of the trapezoid $ABCD$ is therefore

$$P = |EF| \cdot 2v = \sqrt{5} \cdot 2\,\frac{2\sqrt{5}}{5} = 4.$$

Conclusion. All considered trapezoids $ABCD$ have the area 4.

B–I–3

We can rewrite the given equation in the following way

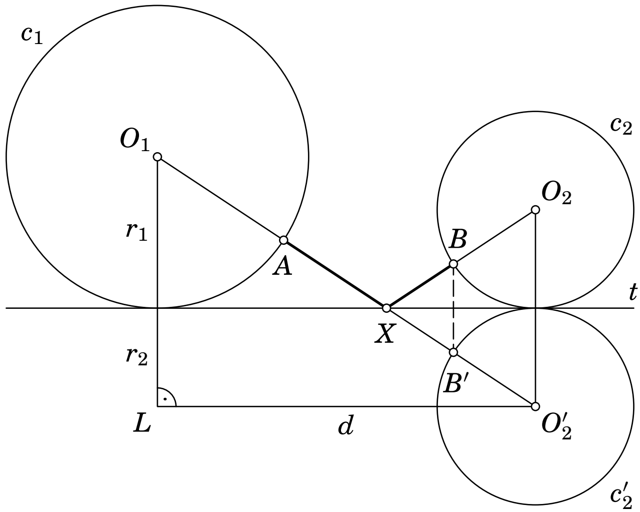
$$2^x + 2^{x+3} = 2^x(1 + 2^3) = 2^x \cdot 3^2 = y^2.$$

If x is an even positive integer, i.e. $x = 2n$ (n is a positive integer), we can see that each pair (x, y) of positive integers in the form $(2n, 2^n \cdot 3)$ is for arbitrary positive integer n a solution of the given equation.

Thus the proof is finished.

B–I–4

We will use a symmetry with respect to the line t. Let us consider a circle c_2' with center O_2' which is a reflection of the circle c_2 in the considered symmetry. The points A and X of a broken line AXB are given as points of intersection of segment O_1O_2' with the circle k_1 and the line t, respectively. A point B id obtained as an image of a



point of intersection B' of the segment O_1O_2' with the circle c_2' in this reflection. Thus we get a point B which lies on the circle c_2. With respect to the construction, the broken line AXB has the smallest possible length.

To determine of its length we can use Pythagoras' theorem in the right-angled triangle O_1LO_2'. The point L lies on a line perpendicular to t going through the center O_1 in the opposite half-plane as the point O_1 in a distance r_2 from t. Thus, for the lengths of its legs we have $|O_1L| = r_1 + r_2$ and $|LO_2'| = d$. It is easy to see, that the smallest length ℓ of the broken line AXB is

$$\ell = |AB'| = \sqrt{(r_1 + r_2)^2 + d^2} - (r_1 + r_2).$$

Category B (Team Competition)

B–T–1

We note that the inequalities $x \geq 2$ and $y \geq 3$ hold, because $\frac{2}{x^2} < 1$ and $\frac{4}{y^2} < 1$.

For $y = 3$, we obtain the equation

$$\frac{2}{x^2} + \frac{1}{x} = \frac{5}{9}.$$

From the above we obtain the equation $5x^2 - 9x - 18 = 0$, which yields $x_1 = -\frac{6}{5}$, which is not positive integer and $x_2 = 3$. We have one solution $x = y = 3$.

We prove that this equation do not other positive integer solutions.

For $y = 4$, we obtain the equation

$$\frac{2}{x^2} + \frac{3}{4x} + \frac{1}{4} = 1,$$

which can be written in the form $3x^2 - 3x - 8 = 0$ with a discriminant $\Delta = 105$ and thus this equation has no solution in positive integers.

For $y \geq 5$, from earlier consideration recall that $x \geq 2$ holds, and the equation has no solution in positive integers, because

$$\frac{2}{x^2} \leq \frac{1}{2} = \frac{50}{100}, \qquad \frac{3}{xy} \leq \frac{3}{10} = \frac{30}{100}, \qquad \frac{4}{y^2} \leq \frac{4}{25} = \frac{16}{100},$$

and therefore

$$\frac{2}{x^2} + \frac{3}{xy} + \frac{4}{y^2} < \frac{96}{100}.$$

The equation has a unique solution in positive integers, namely $x = y = 3$.

Another solution. Since $x \geq 2$ and $y \geq 3$, then $\frac{2}{x^2} < 1$ and $\frac{4}{y^2} < 1$. Further, since

$$\frac{2}{3^2} + \frac{3}{3 \cdot 3} + \frac{4}{3^2} = \frac{2}{9} + \frac{3}{9} + \frac{4}{9} = 1,$$

a solution of the given equation in positive integers is $x = y = 3$.

Assume that there exist some other solution of the equation. Then the first of the unknowns must be smaller and the second one greater than 3 to obtain the sum of fractions equal to 1.

From the inequality $y \geq 3$ the y cannot be smaller than 3. Therefore only x can be smaller, and from inequality $x \geq 2$ only $x = 2$.

For $x = 2$, we have the equation

$$\frac{2}{2^2} + \frac{3}{2y} + \frac{4}{y^2} = 1.$$

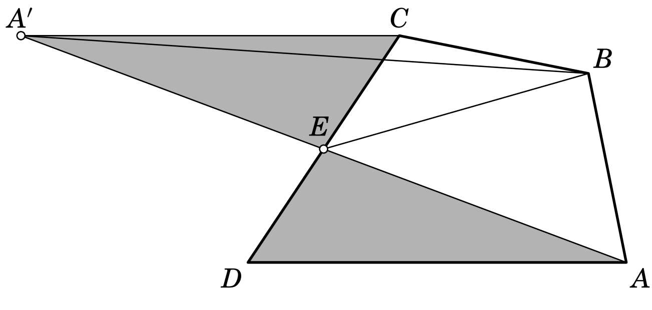
We can rewrite this equation in the form $y^2 - 3y - 8 = 0$ with discriminant $\Delta = 41$. So, there are no positive integer solutions in this case.

It follows that the equation does not have another positive integer solution.

The equation therefore has the unique solution in positive integers $x = y = 3$.

B–T–2

Let A' be a reflection of A with respect to the point E.



It is easy to see that the triangles AED and $A'EC$ are congruent (by the side-angle-side rule). By the assumption, areas of resultant triangles must fulfill the equality

$$S_{BCE} + S_{A'CE} = S_{BCE} + S_{ADE} = S_{ABE} = \tfrac{1}{2} S_{ABCD}.$$

(S_T denotes the area of a polygon T.)

On the other hand, the areas of the triangles AEB and $A'EB$ are equal (both triangles have equal bases AE, $A'E$ and the common altitude). Thus we have

$$S_{A'EB} = S_{BCE} + S_{A'CE},$$

which is true if and only if C lies on the segment $A'B$. This means that $|\angle DAE| = |\angle EA'B|$ and hence the sides BC and AD are parallel.

B–T–3

The given system of equations is equivalent to

$$\begin{aligned} 4ab - 2a + 2b &= -29, \\ 4bc + 2b - 2c &= -11, \\ 4ca + 2c + 2a &= 9 \end{aligned}$$

or

$$\begin{aligned} (2a+1)(2b-1) &= -30, \\ (2b-1)(2c+1) &= -12, \\ (2c+1)(2a+1) &= 10. \end{aligned}$$

Substituting $2a + 1 = x$, $2b - 1 = y$ and $2c + 1 = z$, this is equaivalent to

$$\begin{aligned} xy &= -30, \\ yz &= -12, \\ zx &= 10. \end{aligned}$$

Multiplying these equations yields $(xyz)^2 = 60^2$, and therefore $xyz = \pm 60$. If $xyz = 60$, division yields $x = -5$, $y = 6$ and $z = -2$, which is equivalent to $a = -3$, $b = \frac{7}{2}$ and $c = -\frac{3}{2}$. If $xyz = -60$, we similarly obtain $x = 5$, $y = -6$ and $z = 2$ or $a = 2$, $b = -\frac{5}{2}$ and $c = \frac{1}{2}$. These two tripels are therefore the only solutions of the given system of equations.

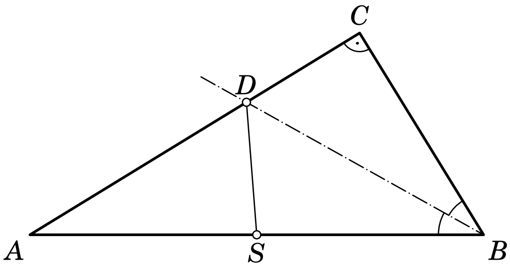
Category C (Individual Competition)

C–I–1

Since the product of the digits is odd, each of the digits must be odd, and its square is odd in each case. If k is the number of digits in n, it follows that the sum of the squares of the digits of n has the same parity as k, so k is even. This is in contradiction with $k = 2011$.

C–I–2

Let S denote the mid-point of the hypotenuse AB (see picture). Then the triangles ASD, BSD and BCD have the same area, which is equal



to $\frac{1}{3}$ of the area of the right-angled triangle ABC (the line segment DS is the median in the triangle ABD and areas of the triangles ABD and BCD are in the ratio $2 : 1$ using conditions of the given problem). Therefore altitudes from vertices S and C in the triangles BSD and BCD are equal (these triangles have the common side BD). Since angles at B in both considered triangles are equal, the triangles BSD and BCD are also congruent. Thus $|BS| = |BC|$ and $|AB| : |BC| = 2 : 1$.

Conclusion. This yields, that measures of angles at A and B in the right-angled triangle ABC are 30° and 60°, respectively.

C–I–3

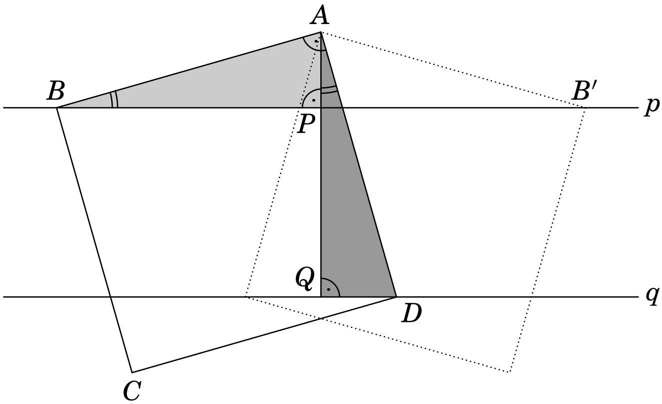
Since a considered ten-digit number n is divisible by 4, the last two digits of this number can be 12 (in this order) only. For each of eight other position of this number (in decimal system) we have always two possibilies (digit 1 or digit 2), i.e. together $2^8 \cdot 1 = 2^8$ possibilities.

Conclusion. There exist $2^8 = 256$ ten-digit numbers with the given property.

C–I–4

Let us assume, that line p is between A and q. Otherwise we can exchange B and D.

Let r be perpendicular to p going through A and let P and Q be points of intersection of r with p and q respectively. Let $ABCD$ be considered square (see the picture).



It is easy to see that the triangles ABP and DAQ are congruent right-angled triangles (they have congruent angles and congruent hypotenuses).

This implies a construction. We draw the perpendicular from the point A to p and q and find its feet P and Q. A point B is on the line p in distance $|AQ|$ from the point P and a point D is on the line q in distance $|AP|$ from the point Q in the opposite half-plane to the half-plane AQB.

The problem has two solutions which are symmetrical to the line r.

Remark. Another solution is based on a rotation. Point D is the image of the point B in rotation around the point A for the angle 90°. So the point D is the intersection point of the line q and the line p' which is image of the p in rotation of p around A by 90°. Since we can rotate clockwise or counter-clockwise, there are two such points D, and we can easily construct the square $ABCD$ from its side AD.

Category C (Team Competition)

C–T–1

We first note that the given equation $(x+y)^2 = 109+xy$ is equivalent to $x^2+xy+y^2 = 109$. Assuming without loss of generality that $x \le y$ holds, we see that $3x^2 \le x^2+xy+y^2 = 109$ must hold, and therefore $x^2 \le \frac{109}{3} < 37$. Since x is a positive integer, x can only be equal to $1,2,3,4,5$ or 6. For $x = 1,2,3,4$ and 6, the equation $x^2+xy+y^2 = 109$ reduces to $y^2+y-108 = 0$, $y^2+2y-105 = 0$, $y^2+3y-100 = 0$, $y^2+4y-93 = 0$ and $y^2+6y-73 = 0$, respectively, none of which has integer solutions. Only for $x = 5$ do we obtain $y^2+5y-89 = 0 \iff (y+12)(y-7) = 0$, which yields the solution $(5,7)$.

Since the equation is symmetric, we obtain the set of all solutions as $\{(5,7),(7,5)\}$.

C–T–2

We have two possibilities for a locus of the point D (see two pictures below).

In the case of the left figure we obtain by double-counting of the length y from the Pythagoras' formula for the right-angled triangles ABF and AFC

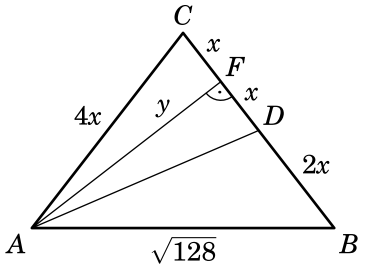
$$128-(3x)^2 = y^2 = (4x)^2 - x^2.$$

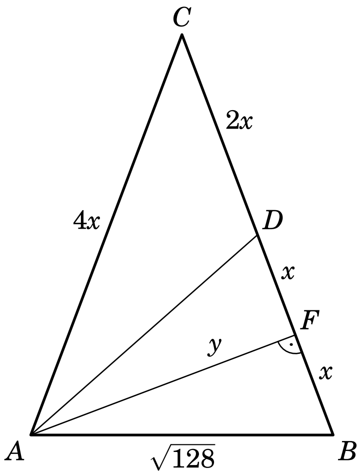
From this we obtain $24x^2 = 128$, and thus $x = \frac{4\sqrt{3}}{3}$. Then $y^2 = 15x^2 = 15\cdot\frac{16}{3} = 80$, and $y = 4\sqrt{5}$. For the area S_1 of this triangle

$$S_1 = \frac{1}{2}\cdot 4x\cdot y = 2xy = 2\cdot\frac{4\sqrt{3}}{3}\cdot 4\sqrt{5} = \frac{32}{3}\sqrt{15}$$

holds. For the perimeter P_1 we have

$$P_1 = 8x+\sqrt{128} = 8\cdot\frac{4\sqrt{3}}{3}+8\sqrt{2} = 8\cdot\left(\frac{4\sqrt{3}}{3}+\sqrt{2}\right).$$





In the case of the right figure we obtain by double-counting of the length y from the Pythagoras' formula applied to the triangles ABF and AFC

$$(4x)^2 - (3x)^2 = y^2 = 128 - x^2.$$

We therefore have $8x^2 = 128$, and thus $x = 4$. Then $y^2 = 7x^2 = 7 \cdot 16$, and $y = 4\sqrt{7}$. For the area S_2 of this triangle

$$S_2 = \frac{1}{2} \cdot 4x \cdot y = 2xy = 2 \cdot 4 \cdot 4\sqrt{7} = 32\sqrt{7}$$

holds. For the perimeter P_2 we have

$$P_2 = 8x + \sqrt{128} = 8 \cdot 4 + 8\sqrt{2} = 8 \cdot \left(4 + \sqrt{2}\right).$$

C–T–3

Rewriting $n^3 - n = (n-1)n(n+1)$ we can see that each of the considered numbers is a product of three consecutive non-negative integers. Since $48 = 2^4 \cdot 3$ we need to find all positive integers n such that $(n-1)n(n+1)$ is divisible by two coprime numbers 2^4 and 3. Since one of each three consecutive integers is always divisible by 3, we must find all positive integers n, such that $(n-1)n(n+1)$ is divisible by $2^4 = 16$. We have two possibilities for the parity of n:

- ▷ n is even. Then $n-1$ and $n+1$ are odd and therefore n must be in the form $n = 16k$ (k is a positive integer).
- ▷ n is odd. Then $n-1$ and $n+1$ are even. Thus $n-1$ or $n+1$ must be divisible by 8 (both of these numbers can't be simultaneously divisible by 4), i.e. $n = 8p + 1$ or $n = 8q - 1$ (p, q are positive integers).

Conclusion. The requested numbers n are in the form $n = 16k$ or $n = 8p + 1$ or $n = 8q - 1$, in which k, p, q are positive integers.

Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Jakub Solovský	GMK Bílovec	8	8	8	8	**32**
	Martin Unger	BRG Kepler Graz	8	8	8	8	**32**
	Tomasz Cieśla	I LO Chorzów	8	8	8	8	**32**
4.	Karel Beneš	GJŠ Přerov	1	8	8	0	**17**
5.	Pavel Francírek	GJŠ Přerov	0	2	3	8	**13**
6.	Josef Malík	GMK Bílovec	1	8	0	0	**9**
	Eva Gocníková	GJŠ Přerov	1	0	0	8	**9**
8.	Fabio De Rubeis	LSS Labriola Roma	0	0	0	8	**8**
9.	Pavel Trutman	GMK Bílovec	3	0	0	0	**3**
10.	Jakub Jaroš	GMK Bílovec	1	0	0	0	**1**
	Andreas Weiss	BRG Kepler Graz	1	0	0	0	**1**
	Marton Liziczai	BRG Kepler Graz	1	0	0	0	**1**
	Artur Koziarz	I LO Chorzów	0	1	0	0	**1**
	Tomasz Depta	I LO Chorzów	1	0	0	0	**1**
	Marek Raclavský	GJŠ Přerov	1	0	0	0	**1**
	Federico Parisi	LSS Labriola Roma	1	0	0	0	**1**
	Matteo Almanza	LSS Labriola Roma	1	0	0	0	**1**
18.	Aleksandra Orłowska	I LO Chorzów	0	0	0	0	**0**
	Renato Catello	LSS Labriola Roma	0	0	0	0	**0**

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Lukáš Chmela	GJŠ Přerov	8	8	8	8	**32**
2.	Jarosław Socha	I LO Chorzów	8	8	8	1	**25**
3.	Václav Kapsia	GMK Bílovec	8	8	8	0	**24**
	Łukasz Ławniczak	I LO Chorzów	8	8	8	0	**24**
5.	Clemens Andritsch	BRG Kepler Graz	8	5	8	1	**22**
6.	Adam Spyra	I LO Chorzów	8	8	5	0	**21**
7.	Jan Krejčí	GMK Bílovec	8	4	8	0	**20**
8.	Bernd Prach	BRG Kepler Graz	8	2	8	0	**18**
9.	Heinz Prach	BRG Kepler Graz	8	1	8	0	**17**
10.	Kateřina Solovská	GMK Bílovec	8	0	8	0	**16**
	Matteo Budoni	LSS Labriola Roma	1	7	8	0	**16**
12.	Ivana Pumprlová	GJŠ Přerov	7	8	0	0	**15**
13.	Michal Šrůtek	GMK Bílovec	8	0	6	0	**14**
14.	Marianna Bastianelli	LSS Labriola Roma	1	0	8	0	**9**
15.	Dominik Nop	GJŠ Přerov	8	0	0	0	**8**
	Michele Tobia	LSS Labriola Roma	0	8	0	0	**8**
17.	Felix Feistritzer	BRG Kepler Graz	6	0	0	0	**6**
	Dario Mostarda	LSS Labriola Roma	6	0	0	0	**6**
19.	Zuzana Gocníková	GJŠ Přerov	1	0	0	0	**1**

Category C (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Igor Lechowski	I LO Chorzów	8	8	8	2	**26**
2.	Daniele Cappuccio	LSS Labriola Roma	8	2	8	2	**20**
3.	Anna Skipirzepa	I LO Chorzów	8	0	8	0	**16**
	Sebastian Borówka	I LO Chorzów	8	0	8	0	**16**
	Tomáš Kremel	GJŠ Přerov	8	0	8	0	**16**
	Agostina Calabrese	LSS Labriola Roma	8	0	8	0	**16**
7.	Jan Gocník	GJŠ Přerov	7	0	8	0	**15**
8.	Jan Dzian	GMK Bílovec	4	0	8	2	**14**
	Tomasz Kasprzak	I LO Chorzów	6	0	8	0	**14**
	Marco Carrozza	LSS Labriola Roma	8	0	6	0	**14**
11.	Tereza Tížková	GMK Bílovec	7	0	6	0	**13**
12.	Marian Poljak	GJŠ Přerov	4	0	8	0	**12**
13.	Šimon Čáp	GMK Bílovec	8	1	2	0	**11**
	Gerda Prach	BRG Kepler Graz	3	0	8	0	**11**
15.	Benedikt Andritsch	BRG Kepler Graz	8	0	2	0	**10**
16.	Martina De Pretis	LSS Labriola Roma	2	2	2	0	**6**
17.	Doris Prach	BRG Kepler Graz	0	0	2	0	**2**
18.	Vojtěch Dorňák	GMK Bílovec	0	0	1	0	**1**

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	I LO Chorzów	7	8	8	**23**
2.	BRG Kepler Graz	2	8	8	**18**
3.	GMK Bílovec	2	8	2	**12**
4.	LSS Labriola Roma	2	8	1	**11**
5.	GJŠ Přerov	2	7	1	**10**

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	BRG Kepler Graz	8	7	2	**17**
2.	LSS Labriola Roma	1	0	4	**5**
3.	I LO Chorzów	2	2	0	**4**
4.	GMK Bílovec	0	2	0	**2**
5.	GJŠ Přerov	1	0	0	**1**

Category C (Team Competition)

Rank	School	1	2	3	∑
1.	I LO Chorzów	2	4	1	**7**
2.	GMK Bílovec	2	0	1	**3**
2.	BRG Kepler Graz	2	0	1	**3**
4.	LSS Labriola Roma	2	0	0	**2**
5.	GJŠ Přerov	1	0	0	**1**

RNDr. Jaroslav Švrček, CSc.
RNDr. Pavel Calábek, Ph.D.
Dr. Robert Geretschläger
Dr. Józef Kalinowski
Dr. Jacek Uryga

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