Palacký University Olomouc, Faculty of Science

MATHEMATICAL DUEL ’13

Jaroslav Švrček
Pavel Calábek
Robert Geretschläger
Józef Kalinowski
Jacek Uryga

Palacký University Olomouc, Faculty of Science

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Preface

The 21st International Mathematical Duel was held from 10–13 March 2013 in Graz. In this year the competition was organized by Bundesrealgymnasium Kepler in Graz.

Five school-teams from Austria, Czech Republic and Poland took part in this traditional mathematical competition, namely from Bundesrealgymnasium Kepler, Graz, Gymnázium M. Koperníka, Bílovec, I Liceum Ogólnokształcące im. J. Słowackiego, Chorzów, Gymnázium J. Škody, Přerov, as well as guest team called All-Stars Graz, made up of students from three schools in Graz.

As usual the competition was provided in the three categories (A – contestants of the last two years, B – contestants of the 5th and 6th years, and C – contestants of the 3rd and 4th years of eight-year grammar school). Twelve contestants (more precisely 4 in any category) of any school took part in this competition, i.e. 60 contestants in total.

This booklet contains all problems with solutions and results of the 21st International Mathematical Duel from the year 2013.

Authors

Problems

Category A (Individual Competition)

A–I–1

Let a be an arbitrary real number. Prove that real numbers b and c certainly exist, such that

$$\sqrt{a^2 + b^2 + c^2} = a + b + c$$

holds.

Jacek Uryga

A–I–2

Let us denote $\mathbb{R}^+ = (0; +\infty)$. Determine all functions $f\colon \mathbb{R}^+ \to \mathbb{R}$, such that

$$xf(x) = xf\left(\frac{x}{y}\right) + yf(y)$$

holds for all positive real values of x and y.

Pavel Calábek

A–I–3

Let O be the circumcenter of an acute-angled triangle ABC. Let D be the foot of the altitude from A to the side BC. Prove that the angle bisector of $\angle CAB$ is also the bisector of $\angle DAO$.

Erich Windischbacher

A–I–4

Let α, β, γ be the interior angles of an obtuse-angled triangle with $\gamma > 90^\circ$. Prove that the inequality

$$\tan\alpha \tan\beta < 1$$

holds.

Józef Kalinowski

Category A (Team Competition)

A–T–1

We are given the following system of equations:

$$x + y + z = a$$
$$x^2 + y^2 + z^2 = b^2$$

with real parameters a and b. Prove that the system of equations has a real solution if and only if the inequality

$$|a| \le |b|\sqrt{3}$$

holds.

Jaroslav Švrček

A–T–2

We are given positive real numbers x, y, z, u with $xyzu = 1$. Prove

$$\frac{x^3}{y^3} + \frac{y^3}{z^3} + \frac{z^3}{u^3} + \frac{u^3}{x^3} \ge x^2 + y^2 + z^2 + u^2.$$

Pavel Calábek

A–T–3

We call positive integers that are written in decimal notation using only the digits 1 and 2 *Graz numbers*. Note that 2 is a 1-digit Graz number divisible by 2^1, 12 is a 2-digit Graz number divisible by 2^2 and 112 is a 3-digit Graz number divisible by 2^3.

a) Determine the smallest 4-digit Graz number divisible by 2^4.
b) Determine an n-digit Graz number divisible by 2^n for $n > 4$.
c) Prove that there must always exist an n-digit Graz number divisible by 2^n for any positive integer n.

Robert Geretschläger

Category B (Individual Competition)

B–I–1

a) Determine all positive integers n, such that the number

$$n^4 + 2n^3 + 2n^2 + 2n + 1$$

is a prime.

b) Determine all positive integers n, such that the number

$$n^4 + 2n^3 + 3n^2 + 2n + 1$$

is a prime.

Jaroslav Švrček

B–I–2

Two circles c_1 and c_2 with radii r_1 and r_2 respectively ($r_1 > r_2$) are externally tangent in point C. A common external tangent t of the two circles is tangent to c_1 in A and to c_2 in B. The common tangent of the two circles in C intersects t in the midpoint of AB. Determine the lengths of the sides of triangle ABC in terms of r_1 and r_2.

Józef Kalinowski

B–I–3

Let s_n denote the sum of the digits of a positive integer n. Determine whether there are infinitely many integers that cannot be represented in the form $n \cdot s_n$.

Jacek Uryga

B–I–4

We call a number that is written using only the digit 1 in decimal notation a *onesy* number, and a number using only the digit 7 in decimal notation a *sevensy* number. Determine a onesy number divisible by 7 and prove that for any sevensy number k, there always exists a onesy number m such that m is a multiple of k.

Robert Geretschläger

Category B (Team Competition)

B–T–1

Two lines p and q intersect in a point V. The line p is tangent to a circle k in the point A. The line q intersects k in the points B and C. The angle bisector of $\angle AVB$ intersects the segments AB and AC in the points K and L respectively. Prove that the triangle KLA is isosceles.

Jaroslav Švrček

B–T–2

Determine all integer solutions of the equation

$$\frac{2}{x} + \frac{3}{y} = 1.$$

Józef Kalinowski

B–T–3

We are given a function $f\colon \mathbb{R} \to \mathbb{R}$, such that $f(m+n) = f(m)f(n)$ holds for all real values of m and n. Furthermore, we know that $f(8) = 6561$.

a) Prove that there exists exactly one real k such that $f(k) = \frac{1}{3}$ and determine the value of k.
b) Prove that no real number ℓ exists, such that $f(\ell) = -\frac{1}{3}$ holds.

Robert Geretschläger

Category C (Individual Competition)

C–I–1

Let $ABCD$ be a parallelogram. The circle c with diameter AB passes through the midpoint of the side CD and through the point D. Determine the measure of the angle $\angle ABC$.

Jaroslav Švrček

C–I–2

Let n be a positive integer. Prove that the number 10^n can always be written as the sum of the squares of two different positive integers.

Jacek Uryga

C–I–3

Joe is travelling by train at a constant speed v. Every time the train passes over a weld seam in the tracks, he hears a click. The weld seams are always exactly 15 m apart. If Joe counts the number of clicks, how many seconds must he count until the number of clicks is equal to the speed of the train in km/h?

Robert Geretschläger

C–I–4

Determine all 3-digit numbers that are exactly 34 times as large as the sum of their digits.

Robert Geretschläger

Category C (Team Competition)

C–T–1

In a triangle ABC with $|AB| = 21$ and $|AC| = 20$, points D and E are chosen on segments AB and AC, respectively, with $|AD| = 10$ and $|AE| = 8$. We find that AC is perpendicular to DE. Calculate the length of BC.

Robert Geretschläger

C–T–2

We consider positive integers that are written in decimal notation using only one digit (possibly more than once), and call such numbers *uni-digit numbers*.

a) Determine a uni-digit number written with only the digit 7 that is divisible by 3.
b) Determine a uni-digit number written with only the digit 3 that is divisible by 7.
c) Determine a uni-digit number written with only the digit 5 that is divisible by 7.
d) Prove that there cannot exist a uni-digit number written with only the digit 7 that is divisible by 5.

Robert Geretschläger

C–T–3

We are given a circle c_1 with midpoint M_1 and radius r_1 and a second circle c_2 with midpoint M_2 and radius r_2. A line t_1 through M_1 is tangent to c_2 in P_2 and a line t_2 through M_2 is tangent to c_1 in P_1. The line t_1 intersects c_1 in a point Q_1 and the line t_2 intersects c_2 in a point Q_2 in such a way that the points P_1, P_2, Q_1 and Q_2 all lie on the same side of M_1M_2. Prove that the lines M_1M_2 and Q_1Q_2 are parallel.

Robert Geretschläger

Solutions

Category A (Individual Competition)

A–I–1

The equality

$$\sqrt{a^2+b^2+c^2} = a+b+c$$

is equivalent to the conditions

$$a^2+b^2+c^2 = (a+b+c)^2 \qquad \text{and} \qquad a+b+c \geq 0, \tag{1}$$

and the first one can be expressed as

$$ab+bc+ca = 0.$$

Now, let a be an arbitrary real number. Let us choose $b \neq 0$ and c such that

$$a+b > 0 \qquad \text{and} \qquad c = -\frac{ab}{a+b}.$$

The last equality yields $ab+bc+ca = 0$ and

$$a+b+c = a+b-\frac{ab}{a+b} = \frac{a^2+ab+b^2}{a+b}.$$

Since the discriminant Δ of the trinomial x^2+bx+b^2 is equal to $-3b^2 < 0$, so for $x = a$ we get $a^2+ab+b^2 > 0$.

Thus we proved that the for an arbitrary a there exist b and c that fulfill the conditions (1).

Another solution. For arbitrary $a \geq 0$ one can take $b = c = 0$ and for arbitrary $a < 0$ the given equality is fulfilled for $b = c = -2a$.

A–I–2

Let t be an arbitrary positive real number. For $x = ty$ we obtain

$$tyf(ty) = tyf(t) + yf(y).$$

Rewriting this equation we get

$$f(ty) = f(t) + \frac{f(y)}{t}.$$

for arbitrary positive t and y. Exchanging t and y we have

$$f(yt) = f(y) + \frac{f(t)}{y}.$$

Comparing the right sides of the last two equations we get

$$f(t)\left(1-\frac{1}{y}\right) = f(y)\left(1-\frac{1}{t}\right).$$

Substituting $y = 2$ with notation $a = 2f(2) \in \mathbb{R}$ we obtain

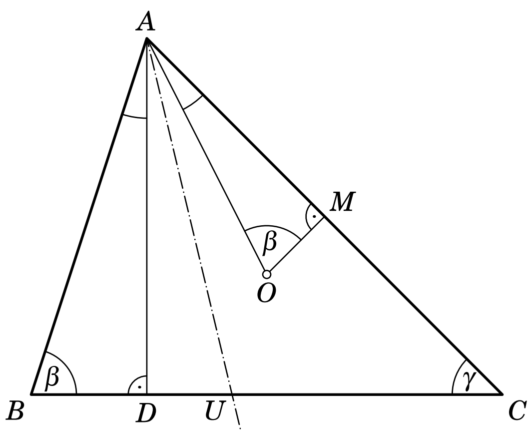
$$f(t) = 2f(2)\left(1-\frac{1}{t}\right) = a\left(1-\frac{1}{t}\right).$$

After easy checking we can see that the function $f(x) = a\left(1-\frac{1}{x}\right)$ satisfies the given equation for arbitrary real a.

A–I–3

Without loss of generality we can assume that $\beta \geq \gamma$ (see the picture). Let M be the midpoint of the side AC of the triangle ABC and U the point of intersection of the angle bisector at A with the side BC. From the picture we can see that

$$|\angle AOM| = |\angle ABD| = |\angle ABC| = \beta.$$



Since ABD and AOM are similar triangles (the angle $\angle ABC$ is equal to half of the angle $\angle AOC$ in the circumcircle of ABC), $|\angle BAD| = |\angle OAM|$ follows. Thus $|\angle DAU| = |\angle OAU|$.

Therefore AU is also the angle bisector of DAO and the proof is finished.

A–I–4

Firstly, we can see that $\alpha+\beta = 180° - \gamma < 90°$ and thus $\tan(\alpha+\beta) > 0$. Using the well-known formula we have

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Since the numerator of the fraction on the right side is evidently a positive real number, the denominator of the same fraction must be positive as well. Therefore it follows that

$$1 - \tan\alpha\tan\beta > 0, \qquad \text{i.e.} \qquad \tan\alpha\tan\beta < 1,$$

and the proof is complete.

Another solution. For $\alpha + \beta < 90°$ we have $\cos(\alpha+\beta) > 0$. Using the well-known formula we further get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta > 0$$

and thus

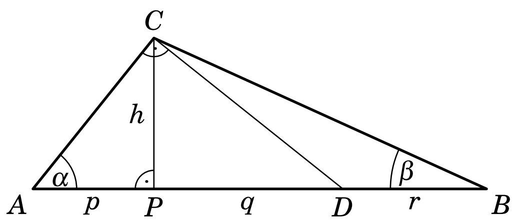
$$\cos\alpha\cos\beta > \sin\alpha\sin\beta.$$

Since $\cos\alpha\cos\beta \neq 0$ we obtain after a short manupulation

$$\tan\alpha \cdot \tan\beta = \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\sin\beta}{\cos\beta} < 1$$

which proves the given inequality.

Another solution. Let CP denote the altitude from C and h its length. Let D be a point on the side longest AB, such that $|\angle ACD| = 90°$. Further, we denote $|AP| = p$, $|PD| = q$ and $|DB| = r$ (see picture).

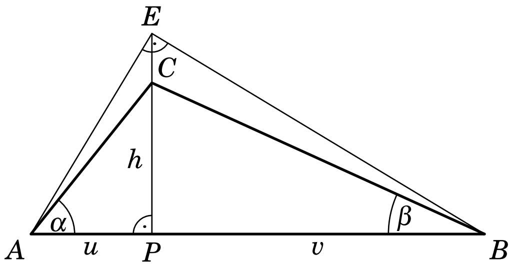


We then obtain

$$\tan\alpha \cdot \tan\beta = \frac{h}{p} \cdot \frac{h}{q+r} < \frac{h}{p} \cdot \frac{h}{q} = \frac{h^2}{pq} = 1,$$

which completes the proof.

Another solution. As in the previous solution we will consider the altitude CP of the length h. On the ray PC we can choose a point E, such that ABE is a right-angled triangle with hypotenuse AB (see picture). Finally, let us write $|PE| = w > h$, $|AP| = u$ and $|PD| = v$.



We then obtain the following estimate

$$\tan\alpha \cdot \tan\beta = \frac{h}{u} \cdot \frac{h}{v} < \frac{w}{u} \cdot \frac{w}{v} = \frac{w^2}{uv} = 1$$

and the proof is finished.

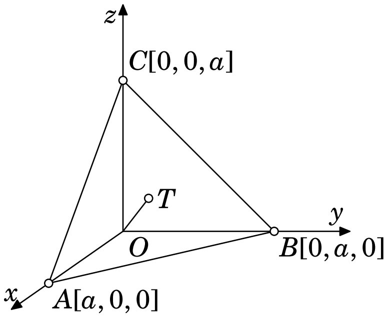
Category A (Team Competition)

A–T–1

Firstly, we will assume that $a \geq 0$. The first equation $x + y + z = a$ with a real parameter a is an analytical equation of the plane which contains the points $A[a, 0, 0]$, $B[0, a, 0]$ and $C[0, 0, a]$ (see the picture) in the Cartesian system $Oxyz$ with the origin at the point $O[0, 0, 0]$. Therefore $ABCO$ is a tetrahedron with edges

$$|AB| = |BC| = |CA| = a\sqrt{2} \qquad \text{and} \qquad |AO| = |BO| = |CO| = a.$$

Similarly the second equation $x^2 + y^2 + z^2 = b^2$ with a real parameter b is an analytical equation of a sphere with the center in O with the radius $|b|$. Similarly for $a < 0$.



Let T be the centroid of the face ABC of the tetrahedron $ABCO$ with $|OT| = d$. It is easy to see that the segment OT is the altitude of this tetrahedron from the vertex O. By double counting we can compute the volume V of the tetrahedron $ABCO$. We have

$$V = \frac{1}{6}a^3 = \frac{1}{3}P \cdot d,$$

where P is the area of the face ABC. After easy manipulation we get

$$P = \frac{1}{2}a\sqrt{2} \cdot a\sqrt{\frac{3}{2}} = \frac{1}{2}a^2\sqrt{3}$$

and thus

$$V = \frac{1}{6}a^3 = \frac{1}{6}a^2\sqrt{3} \cdot d,$$

which implies

$$d = \frac{\sqrt{3}}{3}a.$$

Finally, the given system of equations with unknowns x, y, z (and real parameters a, b) has a real solution if and only if the inequality $d \le |b|$ holds, i.e.

$$\frac{\sqrt{3}}{3}|a| \le |b|.$$

The last inequality is equivalent to $|a| \le |b|\sqrt{3}$ which concludes the proof.

Another solution. Let a triple (x_1, x_2, x_3) of real numbers be a solution of the given system of equations. Using the Cauchy-Schwarz inequality we get

$$3b^2 = (1^2 + 1^2 + 1^2)(x_1^2 + x_2^2 + x_3^2) \ge (x_1 + x_2 + x_3)^2 = a^2, \tag{2}$$

i.e. $a^2 \le 3b^2$ and therefore $|a| \le |b|\sqrt{3}$.

Conversely we can assume that $|a| \le |b|\sqrt{3}$. This inequality implies that there exist real numbers x_1, x_2, x_3 fulfilling the given system of equations (by the inequality (2)) and the proof is finished.

A–T–2

Using the AM-GM inequality for six positive numbers $\frac{x^3}{y^3}, \frac{x^3}{y^3}, \frac{x^3}{y^3}, \frac{y^3}{z^3}, \frac{y^3}{z^3}, \frac{z^3}{u^3}$ we have

$$\frac{1}{6}\left(3 \cdot \frac{x^3}{y^3} + 2 \cdot \frac{y^3}{z^3} + \frac{z^3}{u^3}\right) \ge \sqrt[6]{\frac{x^9}{y^3z^3u^3}} = \sqrt[6]{x^{12}} = x^2.$$

Cyclically we also obtain

$$\frac{1}{6}\left(3 \cdot \frac{y^3}{z^3} + 2 \cdot \frac{z^3}{u^3} + \frac{u^3}{x^3}\right) \ge y^2,$$
$$\frac{1}{6}\left(3 \cdot \frac{z^3}{u^3} + 2 \cdot \frac{u^3}{x^3} + \frac{x^3}{y^3}\right) \ge z^2,$$
$$\frac{1}{6}\left(3 \cdot \frac{u^3}{x^3} + 2 \cdot \frac{x^3}{y^3} + \frac{y^3}{z^3}\right) \ge u^2.$$

Adding up all four inequalities we obtain the required inequality.

Remark: For the proof we can also use the rearrangement inequality for the quadruples

$$\left(\sqrt{\frac{x^3}{y^3}}, \sqrt{\frac{y^3}{z^3}}, \sqrt{\frac{z^3}{u^3}}, \sqrt{\frac{u^3}{x^3}}\right), \ \left(\frac{x}{y}, \frac{y}{z}, \frac{z}{u}, \frac{u}{x}\right), \ \left(\sqrt{\frac{x}{y}}, \sqrt{\frac{y}{z}}, \sqrt{\frac{z}{u}}, \sqrt{\frac{u}{x}}\right).$$

A–T–3

We can prove by induction that there in fact exists a unique n-digit Graz number for any positive integer n. Obviously only 2 is a 1-digit Graz number, as 1 is not divisible by 2^1, but 2 is. We can therefore assume that there exists a unique k-digit Graz number g for some $k \geq 1$. Since g is divisible by 2^k, either $g \equiv 0 \pmod{2^{k+1}}$ or $g \equiv 2^k \pmod{2^{k+1}}$ must hold. Since $10^k \equiv 2^k \pmod{2^{k+1}}$ and $2 \cdot 10^k \equiv 0 \pmod{2^{k+1}}$, we have either $10^k + g \equiv 0 \pmod{2^{k+1}}$ or $2 \cdot 10^k + g \equiv 0 \pmod{2^{k+1}}$, and therefore the unique existence of an $n-1$-digit Graz number.

Now that we know this, it is easy to complete the solution. Since 112 is the 3-digit Graz number, and $112 = 16 \cdot 7$ is divisible by 16, 2112 is the 4-digit Graz number. Since $2112 = 32 \cdot 66$ is divisible by $2^5 = 32$, 22112 is the 5-digit Graz number, and the solution is complete.

Category B (Individual Competition)

B–I–1

a) The given expression can be factorized by the following way

$$n^4 + 2n^3 + 2n^2 + 2n + 1 = (n^4 + 2n^3 + n^2) + (n^2 + 2n + 1) =$$

$$= n^2(n + 1)^2 + (n + 1)^2 = (n^2 + 1)(n + 1)^2.$$

Since $2 \leq n^2+1 < (n+1)^2$ for each positive integer n, the given expression is a product of two positive integers which are greater than or equal to 2. It therefore follows that the given expression is always a composite number.

b) Similarly to case a), we have

$$n^4 + 2n^3 + 3n^2 + 2n + 1 = (n^2 + n + 1)^2.$$

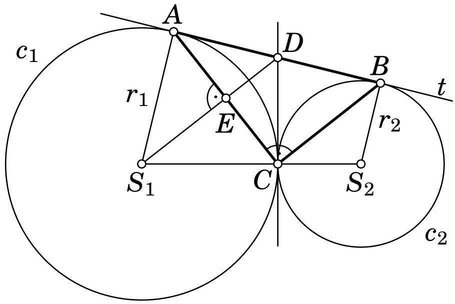
The given expression is the square of a positive integer which is greater than or equal to 3 and therefore there cannot exist a positive integer n such that the given expression is a prime.

B–I–2

Let D be the point of intersection of the common internal tangent of circles c_1 and c_2 in C with tangent t. Since

$$|AD| = |CD| = |BD|$$

we can see that D is the midpoint of the segment AB and simultaneously the center of the Thales circle with diameter AB passing through C. ABC is therefore a right-angled triangle with hypotenuse AB.



It is easy to compute that the length of the hypotenuse AB is $2\sqrt{r_1 r_2}$. Now we can compute the lengths of both legs of the triangle ABC. Using the Pythagorean theorem in the triangle ADS_1 we have

$$|S_1D| = \sqrt{r_1(r_1 + r_2)}.$$

Let E be the midpoint of the segment AC. Using the similarity of right-angled triangles S_1DA and ADE we have

$$\frac{|AE|}{\sqrt{r_1 r_2}} = \frac{r_1}{\sqrt{r_1(r_1 + r_2)}}.$$

Thus

$$|AC| = 2 \cdot |AE| = 2r_1\sqrt{\frac{r_1 r_2}{r_1(r_1 + r_2)}}$$

and analogously

$$|BC| = 2r_2\sqrt{\frac{r_1 r_2}{r_2(r_1 + r_2)}}.$$

Conclusion. In summary, we have obtained

$$|AB| = 2\sqrt{r_1 r_2}, \qquad |AC| = 2r_1\sqrt{\frac{r_1 r_2}{r_1(r_1+r_2)}},$$

and

$$|BC| = 2r_2\sqrt{\frac{r_1 r_2}{r_2(r_1+r_2)}}.$$

B–I–3

Write the number n as

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10a_1 + a_0,$$

where $a_k, a_{k-1}, \ldots, a_0$ are all the digits of n. Thus

$$\begin{aligned} n - s_n &= 10^{k-1} a_{k-1} + \ldots + 10a_1 + a_0 - (a_k + a_{k-1} + \ldots + a_0) = \\ &= (10^k - 1)a_k + (10^{k-1} - 1)a_{k-1} + \ldots + (10-1)a_1. \end{aligned}$$

Note that

$$10^m - 1 = (10-1)(10^{m-1} + 10^{m-2} + \ldots + 10^2 + 10^1 + 1),$$

so the numbers $10^m - 1$ are divisible by 9 for all positive integers m. This implies that the number $n - s_n$ is divisible by 9 and so by 3. Hence we get that both n as well as s_n give the same remainder when divided by 3.

One can easily show that the product of two integers, which give the same remainder when divided by 3, can never give the remainder 2. In fact, for arbitrary integers p and q we have

$$\begin{aligned} (3p)(3q) &= 3(3pq), \\ (3p+1)(3q+1) &= 3(3pq + p + q) + 1, \\ (3p+2)(3q+2) &= 3(3pq + 2p + 2q + 1) + 1. \end{aligned}$$

Therefore no integer of the form $3k + 2$ can be represented as the product of n and s_n. This proves that there are infinitely many integers with the desired property.

Another solution. We know that n is divisible by 3 if and only if s_n is divisible by 3 as well. Let a be a positive integer not divisible by 3 (there exist infinitely many such integers). If the number $3a$ can be expressed as the product $n \cdot s_n$, one of the factors n or s_n is divisible by 3, and so the second one is divisible by 3 too and the product $n \cdot s_n$ is divisible by 9. It follows that a is divisible by 3, which is a contradiction. It follows that the number $3a$ cannot be expressed in the form $n \cdot s_n$.

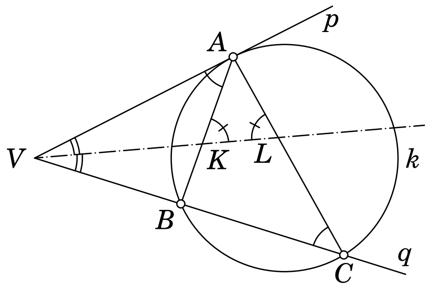
B–I–4

A possible onesy number divisible by seven is given by $111111 = 111 \cdot 1001 = 111 \cdot 7 \cdot 11 \cdot 13$.

In order to see that there always exists a onesy multiple of any sevensy number k, note that there exist an infinite number of onesy numbers. By the Dirichlet principle, there must therefore exist two different onesy numbers $m_1 > m_2$ with $m_1 \equiv m_2 \pmod{k}$. It therefore follows that $m_1 - m_2$ is divisible by k. The number $m_1 - m_2$ can be written as $m_1 - m_2 = m \cdot 10^r$, where m is also a onesy number. Since k is certainly not divible by 2 or 5, it follows that m must also be divisible by k, and the proof is complete.

Category B (Team Competition)

B–T–1



Since p is tangent to the circle k, $|\angle VAB| = |\angle VCA|$ must hold. Since the line KL is the angle bisector of the angle AVC we get

$$|\angle AKL| = |\angle VAK| + |\angle AVK| = |\angle VAB| + |\angle AVK| =$$
$$= |\angle VCA| + |\angle CVL| = |\angle VCL| + |\angle CVL| = |\angle ALV| = |\angle ALK|.$$

This means that AKL is an isosceles triangle with the base KL, which completes the proof.

B–T–2

We can rewrite the given equation in the form

$$xy - 3x - 2y + 6 = 6, \qquad \text{i.e.} \qquad (x-2)(y-3) = 6.$$

The integer 6 can be factored as the product of two integers as follows:

$$6 = 1\cdot 6 = 6\cdot 1 = 2\cdot 3 = 3\cdot 2 = (-1)\cdot(-6) = (-6)\cdot(-1) = (-2)\cdot(-3) = (-3)\cdot(-2).$$

Therefore we have to discuss eight cases in the following table.

$x-2$	1	6	2	3	-1	-6	-2	-3
$y-3$	6	1	3	2	-6	-1	-3	-2
x	3	8	4	5	1	-4	0	-1
y	9	4	6	5	-3	2	0	1

Since $x \neq 0$ and $y \neq 0$, the given equation has exactly seven solutions, namely:

$$(x,y) \in \{(3;9),(8;4),(4;6),(5;5),(1;-3),(-4;2),(-1;1)\}.$$

B–T–3

It is perhaps easiest to first note that there can be no real number ℓ such that $f(\ell) < 0$ holds. If this were the case, we would have $0 > f(\ell) = f\left(\frac{\ell}{2} + \frac{\ell}{2}\right) = f\left(\frac{\ell}{2}\right)^2$, which is not possible. This completes the proof of the part b).

We now note that $f(m) = f(m + 0) = f(m) \cdot f(0)$ implies $f(0) = 1$ and $1 = f(0) = f(-m + m) = f(-m) \cdot f(m)$ implies $f(-m) = \frac{1}{f(m)}$. If $f(m) = f(n)$ for some $m > n$, we have $f(m - n) = f(m) \cdot f(-n) = 1 = f(0)$, and therefore $f(x) = 1$ for any rational multiple of $m - n$, which is clearly not possible if $f(0) = 1$ and $f(8) > 1$.

Since $6561 = f(8) = f(4)^2$, we have $f(4) = 81$, and similarly $f(2) = 9$ and $f(1) = 3$. We therefore have the unique value of $f(-1) = \frac{1}{3}$.

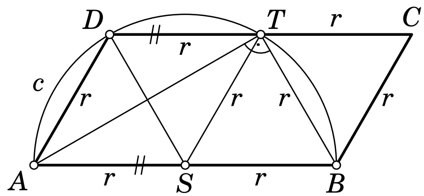
Category C (Individual Competition)

C–I–1

Let S and T denote the midpoints of the sides AB, CD of the given parallelogram respectively, and $2r$ their lengths (see the picture). Since the points D and T lie on the Thales circle with diameter AB, the equalities

$$|SA| = |SB| = |ST| = |AD| = |BC| = r$$

hold. Further, it is easy to see that the chords AB and DT of the circle c are parallel.



The quadrilateral $ABTD$ is therefore an isosceles trapezoid with bases AB, DT and with $|AD| = |BT| = r$. Therefore SBT is an equilateral triangle with sides of the length r which yields

$$|\angle SBT| = |\angle ABT| = 60^\circ.$$

Similarly we can prove that BCT is also an equilateral triangle with sides of the length r, and thus $|\angle ABC| = 120^\circ$.

C–I–2

For $n = 1$ and $n = 2$ we have

$$10^1 = 1^2 + 3^2 \qquad \text{and} \qquad 10^2 = 6^2 + 8^2.$$

Now, observe that in case, if n is a positive odd number, that is, if $n = 2k - 1$ with $k > 0$, then

$$10^n = 10^{n-1} \cdot 10 = 10^{2k-2} \cdot (1^2 + 3^2) = (10^{k-1})^2 + (3 \cdot 10^{k-1})^2$$

and if $n = 2k$ with $k > 0$ is a positive even number, then we have

$$10^n = 10^{n-2} \cdot 10^2 = 10^{2k-2} \cdot (6^2 + 8^2) = (10^{k-1})^2 + (3 \cdot 10^{k-1})^2.$$

This proves our assumption for every positive integer n.

C–I–3

If the train is travelling at v km/h, we know that it is travelling at $\frac{v}{3,6}$ m/sec. This means that it crosses a total of $\frac{v}{3,6} : 15 = \frac{v}{54}$ stretches of 15 m track each second, or exactly v such stretches of 15 m track in 54 seconds. Joe must therefore count for exactly 54 seconds.

C–I–4

A three digit number can be written in the form $100a + 10b + c$. The sum of the number's digits is $a + b + c$, and any number with the required property must therefore also have the property

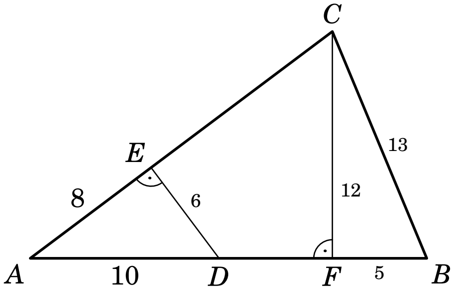
$$100a + 10b + c = 34(a + b + c) \iff 66a - 33c = 24b.$$

Dividing by 3, this is equivalent to $11(2a - c) = 8b$. Since the left side of this equation is divisible by 11, and no single digit positive number can be divisible by 11, it follows that b must be equal to 0. In this case, the property is equivalent to $2a = c$, and the numbers fulfilling the requirements are therefore 102, 204, 306 and 408.

Category C (Team Competition)

C–T–1

Let F be the foot of C on AB. In the triangle ADE, it is easy to calculate the length of $|DE| = 6$. Right-angled triangles ADE and ACF are similar since they have a common angle in A, and since $|AC| = 20 = 2 \cdot |AD|$, we have $|CF| = 2 \cdot |DE| = 12$ and $|AF| = 2 \cdot |AE| = 16$. In the right-angled triangle CFB we therefore have the sides $|FC| = 12$ and $|FB| = |AB| - |AF| = 21 - 16 = 5$, and therefore the hypotenuse $|BC| = \sqrt{5^2 + 12^2} = 13$.



C–T–2

a) $777 = 7 \cdot 111 = 7 \cdot 37 \cdot 3$.

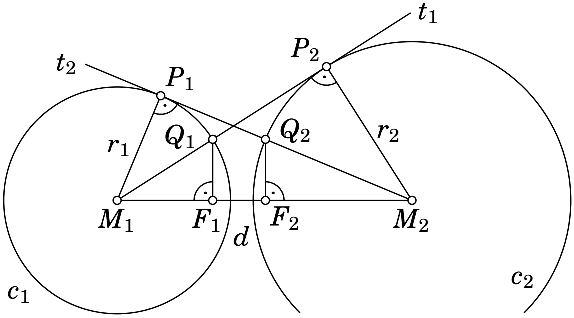
b) $333333 = 333 \cdot 1001 = 333 \cdot 7 \cdot 11 \cdot 13$.

c) $555555 = 555 \cdot 7 \cdot 11 \cdot 13$.

d) The last digit of any number divisible by 5 is always either 0 or 5. Any number that is divisible by 5 can therefore not be written using only the digit 7.

C–T–3

Let F_1 and F_2 be the feet of Q_1 and Q_2 on M_1M_2 respectively, and denote the distance between M_1 and M_2 as d. Right-angled triangles $M_1M_2P_2$ and $M_1Q_1F_1$ are similar, since they have a common angle in M_1. It therefore follows that



$$|Q_1F_1| : |M_1Q_1| = |M_2P_2| : |M_1M_2|$$

holds which is equivalent to $|Q_1F_1| : r_1 = r_2 : d$ or $|Q_1F_1| = r_1r_2/d$. By completely analogous calculation with reversed roles of the circles, we also obtain $|Q_2F_2| = r_1r_2/d$, and since Q_1 and Q_2 are equidistant from M_1M_2, it follows that the lines M_1M_2 and Q_1Q_2 are parallel as claimed.

Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Bernd Prach	Graz Kepler	8	8	8	8	**32**
2.	Łukasz Ławniczak	I.LO Chorzów	8	7	8	8	**31**
3.	Lukáš Knob	GJŠ Přerov	8	3	8	8	**27**
4.	Jarosław Socha	I.LO Chorzów	8	2	8	8	**26**
5.	Clemens Andritsch	Graz Kepler	7	2	8	8	**25**
5.	Petr Vincena	GJŠ Přerov	8	1	8	8	**25**
7.	Petr Vaněk	GMK Bílovec	7	2	7	8	**24**
8.	Markéta Calábková	GJŠ Přerov	4	2	8	8	**22**
9.	Marian Poljak	GJŠ Přerov	8	1	3	8	**20**
10.	Jan Šarman	GMK Bílovec	0	2	8	8	**18**
11.	Artur Minorczyk	I.LO Chorzów	2	3	0	8	**13**
11.	Heinz Prach	Graz Kepler	3	2	0	8	**13**
13.	Jan Krejčí	GMK Bílovec	4	2	0	0	**6**
14.	Michal Šrůtek	GMK Bílovec	1	4	0	0	**5**
15.	Benjamin von Berg	Graz All-Stars	2	2	0	0	**4**
16.	Marek Grabowski	I.LO Chorzów	2	0	0	0	**2**
16.	Hannah Lichtenegger	Graz All-Stars	2	0	0	0	**2**
16.	Viet Anh Nguyen	Graz All-Stars	1	1	0	0	**2**
16.	Andrea Triebl	Graz All-Stars	2	0	0	0	**2**
20.	Martina Svibic	Graz Kepler	0	0	0	0	**0**

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Tomáš Kremel	GJŠ Přerov	0	8	8	2	**18**
2.	Jan Gocník	GJŠ Přerov	4	0	8	2	**14**
3.	Lucie Holušová	GMK Bílovec	4	8	0	0	**12**
4.	Vivian Obernosterer	Graz All-Stars	2	0	7	2	**11**
4.	Gerda Prach	Graz Kepler	0	1	8	2	**11**
4.	Michał Ślusarczyk	I.LO Chorzów	0	1	8	2	**11**
7.	Michal Koupil	GJŠ Přerov	0	0	8	2	**10**
7.	Jakub Paliga	I.LO Chorzów	0	0	8	2	**10**
9.	Michał Osadnik	I.LO Chorzów	0	1	8	0	**9**
10.	Benedikt Andritsch	Graz Kepler	0	6	0	1	**7**
11.	Jiří Andrlík	GJŠ Přerov	0	5	0	0	**5**
11.	Alicja Kalisz	I.LO Chorzów	1	2	0	2	**5**
13.	Zuzana Beigerová	GMK Bílovec	0	0	0	2	**2**
13.	Jiří Grygar	GMK Bílovec	0	0	0	2	**2**
13.	Benjamin Holter	Graz All-Stars	0	0	0	2	**2**
13.	Filip Rescec	Graz All-Stars	0	0	0	2	**2**
17.	Tomáš Moravec	GMK Bílovec	0	1	0	0	**1**
17.	Doris Vogel	Graz All-Stars	0	1	0	0	**1**
19.	Doris Prach	Graz Kepler	0	0	0	0	**0**

Category C (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Marcin Socha	I.LO Chorzów	8	8	8	8	**32**
1.	Marcin Sztuka	I.LO Chorzów	8	8	8	8	**32**
1.	Karol Szydlik	I.LO Chorzów	8	8	8	8	**32**
4.	Jan Równicki	I.LO Chorzów	8	8	7	8	**31**
5.	Bára Tížková	GMK Bílovec	8	6	8	8	**30**
6.	Konstantin Andritsch	Graz Kepler	8	4	7	8	**27**
7.	Daniel Horiatakis	Graz Kepler	8	8	6	4	**26**
8.	Anežka Malčíková	GMK Bílovec	0	7	8	8	**23**
9.	Zdeněk Kroča	GJŠ Přerov	0	0	8	7	**15**
10.	Karolína Vojkůvková	GMK Bílovec	4	7	0	3	**14**
11.	Berenika Čermáková	GMK Bílovec	2	0	0	8	**10**
12.	Denisa Chytilová	GJŠ Přerov	0	6	0	3	**9**
13.	Alexander Kropiunig	Graz All-Stars	0	0	1	7	**8**
14.	Vinzenz Holzner	Graz Kepler	2	0	1	3	**6**
15.	Christian Thallinger	Graz All-Stars	0	0	1	3	**4**
16.	Lukáš Kremel	GJŠ Přerov	0	0	0	3	**3**
17.	Verena Haas	Graz Kepler	0	0	1	0	**1**
17.	Jiří Hanák	GJŠ Přerov	1	0	0	0	**1**
17.	Anja Zotter	Graz All-Stars	0	0	0	1	**1**
20.	Regina Salloker	Graz All-Stars	0	0	0	0	**0**

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	Graz Kepler	8	8	8	**24**
2.	GJŠ Přerov	7	1	8	**16**
3.	I.LO Chorzów	8	0	7	**15**
4.	Graz All-Stars	6	0	8	**14**
5.	GMK Bílovec	0	0	2	**2**

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	Graz Kepler	8	8	0	**16**
2.	GJŠ Přerov	0	5	4	**9**
3.	GMK Bílovec	0	8	0	**8**
4.	Graz All-Stars	0	6	0	**6**
4.	I.LO Chorzów	0	2	4	**6**

Category C (Team Competition)

Rank	School	1	2	3	∑
1.	I.LO Chorzów	8	8	8	**24**
2.	Graz All-Stars	4	7	0	**11**
3.	GMK Bílovec	1	8	0	**9**
3.	GJŠ Přerov	1	8	0	**9**
3.	Graz Kepler	1	8	0	**9**

RNDr. Jaroslav Švrček, CSc.
RNDr. Pavel Calábek, Ph.D.
Dr. Robert Geretschläger
Dr. Józef Kalinowski
Dr. Jacek Uryga

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e-mail: vup@upol.cz

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