Akademicki Zespół Szkół Ogólnokształcących, Chorzów
Uniwersytet Śląski, Katowice

MATHEMATICAL DUEL '15

Jaroslav Švrček
Pavel Calábek
Robert Geretschläger
Gottfried Perz
Józef Kalinowski
Jacek Uryga

Creation of this booklet is co-funded by the Erasmus+ Strategic Partnership "MD_15_17: Mathematical Duel Plus 2015–17" No.2014-1-AT01-KA201-000908.

Olomouc 2015

First edition

ISBN 978-80-244-4650-9

Preface

The 23rd Mathematical Duel was held in Bielsko-Biała from 11th till 15th March 2015. It was the first of three competitions planned as part of the Duel Plus project, which is entirely financed by the Erasmus Plus programme. The students from three age groups—Category A (age 17–19), Category B (age 15–17), Category C (age 13–15)—took part in individual and team competition. The teams came from Austrian, Czech and Polish schools, namely Bundesrealgymnasium Kepler in Graz, Gymnázium Mikuláše Koperníka in Bílovec, Gymnázium Jakuba Škody in Přerov, V Liceum Ogólnokształcące in Bielsko-Biała, Akademicki Zespół Szkół Ogólnokształcących in Chorzów.

Individual competition started on 12th March at 9 a.m. The students in each age group were allowed two hours and thirty minutes to solve four problems (they could achieve maximum score of 8 points for each problem). Since all the problems were formulated in English, the teachers in charge of their teams helped them understand the content. The students could provide solutions either in their native languages or in English.

The second part of the Duel, i.e. team competition started after coffee and snack break. The participating schools were allowed to send 12 persons—four persons for each age group. Thus, four students representing the same school and age group could join one team. The teams were told to solve three problems within 100 minutes.

This booklet contains all problems with solutions and results of the 23rd Mathematical Duel from the year 2015.

Authors

Problems

Category A (Individual Competition)

A–I–1

Find all pairs (a, b) of real numbers such that the roots of the cubic equation

$$x^3 + ax^2 + bx + ab = 0$$

are the numbers $-a$, $-b$ and $-ab$.

Jaroslav Švrček

A–I–2

Prove that for any arbitrary positive integer n there exists a perfect square such that sum of its digits is equal to n^2.

Jacek Uryga

A–I–3

Let u and v be the distances of an arbitrary point of the side AB of the acute-angled triangle ABC to its sides AC and BC. Furthermore let h_a, h_b be the lengths of the altitudes from its vertices A and B, respectively. Prove that the inequalities

$$\min\{h_a, h_b\} \leq u + v \leq \max\{h_a, h_b\}$$

hold.

Józef Kalinowski

A–I–4

The positive integers k, l, m, n fulfil the equation

$$k^2l^2 - m^2n^2 = 2015 + l^2m^2 - k^2n^2.$$

Find all possible values of $k + l + m + n$.

Gottfried Perz

Category A (Team Competition)

A–T–1

Determine all pairs (x, y) of integers fulfilling the equation

$$x^2 - 3x - 4xy - 2y + 4y^2 + 4 = 0.$$

Pavel Calábek

A–T–2

We are given a triangle ABC in the plane. Prove that for any triple u, v, w of positive real numbers there exists a point P inside the triangle ABC such that

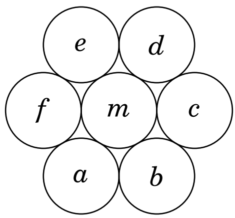
$$S_{ABP} : S_{BCP} : S_{CAP} = u : v : w.$$

[*Remark*. S_{XYZ} denotes the area of a triangle XYZ.]

Jacek Uryga

A–T–3

As shown in the figure, a circle is surrounded by six touching circles of the same size.



A real number a, b, c, d, e, f or m is written in the interior of each circle. It is known that each of these numbers is equal to the product of all numbers in the interiors of the touching circles. Determine all possible values of m and prove that no other value is possible.

Robert Geretschläger

Category B (Individual Competition)

B–I–1

How many triples (a, b, c) of positive integers with

$$abc = 45\,000$$

exist?

Józef Kalinowski

B–I–2

Let ABC be a triangle with right angle at the vertex C. Let ACP and BCQ be right-angled isosceles triangles external to ABC with right angles at P and Q, respectively. Furthermore let F be the foot of the altitude from C to AB and D, E be points of intersection of the line AC with PF and the line BC with QF, respectively. Prove that $|DC| = |EC|$.

Gottfried Perz

B–I–3

Let p, q, r, s be non-negative real numbers with $p \le q \le r \le s$. Prove that the inequality

$$\frac{p+q+r+s}{4} \ge \frac{p+q+r}{3}$$

holds. When does equality hold?

Józef Kalinowski

B–I–4

Let $ABCD$ be a circumscribed quadrilateral with right angles at B and D. Prove that $ABCD$ is a deltoid.

Jaroslav Švrček

Category B (Team Competition)

B–T–1

Determine all 5-tuples (a, b, c, d, e) of positive integers such that each of the fractions

$$\frac{a+b}{c+d},\quad \frac{b+c}{d+e},\quad \frac{c+d}{e+a},\quad \frac{d+e}{a+b},\quad \frac{e+a}{b+c}$$

is an integer.

Jaroslav Švrček

B–T–2

a) Jacek has four sticks of integer length that he puts on the table to form a convex quadrilateral. No matter which three of the four sticks he chooses, there is never any way he can form a triangle. What is the smallest possible circumference of the quadrilateral that Jacek can make?

b) Jozef has six sticks of integer length. He can put them on the table and form a convex hexagon, but just like Jacek, there is never any way he can form a triangle with three of his sticks. What is smallest possible length of the longest of Jozef's sticks?

Robert Geretschläger

B–T–3

Determine the number of all six-digit palindromes which are divisible by seven.

[*Remark*. The six-digit palindrome is a positive integer which is written in the form $\overline{abccba}$ and $a \neq 0$, b, c are digits of the decimal system.]

Pavel Calábek

Category C (Individual Competition)

C–I–1

Determine all pairs (m,n) of integers satisfying the following equation

$$m + \frac{1}{n} = n + \frac{1}{m}.$$

Jaroslav Švrček

C–I–2

Let ABC be an acute-angled triangle with integer angles α, β and γ. We are given that the external angle ε at the vertex A is an integer multiple of α with $\varepsilon = k\alpha$. Determine all possible values of k.

Robert Geretschläger

C–I–3

Determine all triples (a,b,c) of positive integers such that each of the fractions

$$\frac{a+b}{b+c}, \quad \frac{b+c}{c+a}, \quad \frac{c+a}{a+b}$$

is an integer.

Jaroslav Švrček

C–I–4

We are given a rectangle $ABCD$ with $|AB| = 4$ and $|\angle ABD| = 30^\circ$. Point E lies on the circumcircle of $ABCD$ with $CE \parallel BD$. Determine (with proof) the length of the segment AE.

Robert Geretschläger

Category C (Team Competition)

C–T–1

Determine the number of all pairs (x, y) of integers for which the inequality

$$2|x| + 3|y| < 23$$

is fulfilled.

Pavel Calábek

C–T–2

We are given the right-angled triangle ABC with legs $|AC| = 1$ and $|BC| = \sqrt{3}$. Let us consider two circles with diameters AC and BC. Calculate the area of the common part of both circles.

Jacek Uryga

C–T–3

A *wavy number* is a number in which the digits alternately get larger and smaller (or smaller and larger) when read from left to right. (For instance, 3629263 and 84759 are wavy numbers but 45632 is not.)

a) Two five-digit wavy numbers m and n are composed of all digits from 0 to 9. (Note that the first digit of a number cannot be 0.) Determine the smallest possible value of $m + n$.
b) Determine the largest possible wavy number in which no digit occurs twice.
c) Determine a five-digit wavy number that can be expressed in the form $ab + c$, where a, b and c are all three-digit wavy numbers.

Robert Geretschläger

Solutions

Category A (Individual Competition)

A–I–1

Let us suppose that there exist real numbers a, b fulfilling the conditions of the given problem. Using Vièta's formulas we get

$$
\begin{aligned}
-a^2b^2 = -ab \quad &\Leftrightarrow \quad ab(ab-1) = 0, && (1)\\
ab + a^2b + ab^2 = b \quad &\Leftrightarrow \quad ab(1+a+b) = b, && (2)\\
-a - b - ab = -a \quad &\Leftrightarrow \quad b(a+1) = 0, && (3)
\end{aligned}
$$

Further we will consider two different cases for factors on the left side of (1):

- ▷ Let $ab = 0$. If $a = 0$, then by (2) and (3) $b = 0$ also holds. If $b = 0$, then similarly by (2) and (3) a can be arbitrary real number. It means that desired pairs (a, b) of real numbers are $(a, 0)$, where a is any real number. The corresponding cubic equation is in the form $x^3 + ax^2 = 0$, which has three real roots $-a$, 0 and 0.
- ▷ Let $ab = 1$. Since $b \neq 0$, then $a = -1$ must hold by (3). From the initial condition $ab = 1$ we further have $b = -1$. It is easy to check, that the pair $(a, b) = (-1; -1)$ of real numbers also satisfies the relation (2). In this case we will obtain the cubic equation $x^3 - x^2 - x + 1 = 0$, which has three real roots -1, -1 and 1.

Conclusion. The solutions of the given problem are the following pairs of real numbers: $(a, b) = (a, 0)$, where a is any real number, and also $(a, b) = (-1; -1)$.

A–I–2

Let us take an arbitrary increasing sequence $p_1, p_2, \ldots, p_n$ of $n > 0$ positive integers that satisfies an additional condition

$$p_k > 2p_{k-1} \quad \text{for every} \quad 1 < k \le n.$$

We show that the digit sum of the square number

$$N = \left(10^{p_1} + 10^{p_2} + \ldots + 10^{p_n}\right)^2$$

is equal to n^2.

First note that

$$N = \sum_{i,j=1}^{n} 10^{p_i+p_j} = \sum_{i=1}^{n} 10^{2p_i} + \sum_{1\le i<j\le n} 2 \cdot 10^{p_i+p_j}.$$

Every term in both sums is either an integer of the form $100\ldots0$ or an integer of the form $200\ldots0$. The number of 0's at the end of every integer is equal to the exponent of the power of 10 in the respective term.

Note that each exponent in both sums can be expressed as p_i+p_j, with $p_i \le p_j$ (in case $p_i = p_j$ we have $p_i + p_j = 2p_i$). It is easy to see that no two of them are equal. Indeed, taking two such exponents $p_{i_1} + p_{j_1}, p_{i_2} + p_{j_2}$ we can assume that

a) $j_1 < j_2$,
b) $i_1 < i_2$ and $j_1 = j_2$.

In the case

a) we have $p_{i_1} + p_{j_1} \le 2p_{j_1} < p_{j_2} < p_{i_2} + p_{j_2}$,
b) we obviously have $p_{i_1} + p_{j_1} < p_{i_2} + p_{j_2}$.

The first sum has n terms and the second one has $n(n-1)/2$ terms. Thus the number N consists of n one's as well as $n(n-1)/2$ two's and zero's as other digits. So the digit sum of N is equal to $n + n(n-1) = n^2$.

Another solution. From the following equalities

$$\begin{aligned}
(9m)^2 &= 9(9m^2), & (9m+1)^2 &= 9(9m^2+2m)+1,\\
(9m+2)^2 &= 9(9m^2+4m)+4, & (9m+3)^2 &= 9(9m^2+6m+1),\\
(9m+4)^2 &= 9(9m^2+8m+1)+7, & (9m+5)^2 &= 9(9m^2+10m+2)+7,\\
(9m+6)^2 &= 9(9m^2+12m+4), & (9m+7)^2 &= 9(9m^2+14m+5)+4,\\
(9m+8)^2 &= 9(9m^2+16m+7)+1,
\end{aligned}$$

results that for every integer n the perfect square n^2 can be expressed as $9k$, $9k+1$, $9k+4$ or $9k+7$ with k as an integer.

Let us observe that for every $k > 0$ the digit sums of the squares

$$(10^k - 1)^2 = 10^{2k} - 2 \cdot 10^k + 1 = \underbrace{99\ldots9}_{k-1 \text{ digits}}8\underbrace{00\ldots0}_{k-1 \text{ digits}}1$$

$$(10^k - 2)^2 = 10^{2k} - 4 \cdot 10^k + 4 = \underbrace{99\ldots9}_{k-1 \text{ digits}}6\underbrace{00\ldots0}_{k-1 \text{ digits}}4$$

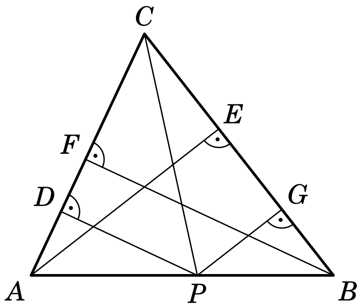
$$(10^k - 3)^2 = 10^{2k} - 6 \cdot 10^k + 9 = \underbrace{99\ldots9}_{k-1 \text{ digits}}4\underbrace{00\ldots0}_{k-1 \text{ digits}}9$$

$$(10^{k+1} - 5)^2 = 10^{2k+2} - 10^{k+2} + 25 = \underbrace{99\ldots9}_{k \text{ digits}}\underbrace{00\ldots0}_{k \text{ digits}}25$$

are equal to $9k$, $9k + 1$, $9k + 4$ and $9k + 7$, respectively. This proves our assertion for $n > 2$. The cases $n = 1, 2$ are obvious.

A–I–3

Let P be an arbitrary point inside the segment AB. Let point $G \in BC$ is such that $PG \perp BC$ and point $D \in AC$ is such that $PD \perp AC$. Then $u = |PD|$ and $v = |PG|$.



Let AE and BF be altitudes of the triangle ABC. Then

$$2S_{ABC} = |BC| \cdot |AE| = |AC| \cdot |BF|,$$

where S_{ABC} denotes the area of the triangle ABC.

Dividing the triangle ABC into two triangles APC and PBC we obtain

$$2S_{ABC} = |AC| \cdot |PD| + |BC| \cdot |PG|.$$

Then the equality

$$|BC| \cdot |AE| = |AC| \cdot |PD| + |BC| \cdot |PG| = |AC| \cdot |BF|$$

holds.

Let us consider two possible cases:

a) $|AC| \le |BC|$. Then by the above equality we have

$$|BC| \cdot |AE| = |AC| \cdot |PD| + |BC| \cdot |PG| \le |BC| \cdot |PD| + |BC| \cdot |PG|,$$

whence $|AE| \le |PD| + |PG|$. Also by the equality in the case we have

$$|AC| \cdot |BF| = |AC| \cdot |PD| + |BC| \cdot |PG| \ge |AC| \cdot |PD| + |AC| \cdot |PG|,$$

thus $|BF| \ge |PD| + |PG|$ and therefore

$$|AE| \le |PD| + |PG| \le |BF|.$$

In case a), because $|AC| \le |BC|$, the altitudes AE and FB fulfil the inequality $|FB| \ge |AE|$. Then we have

$$\min\{h_a, h_b\} = |AE| \quad \text{and} \quad \max\{h_a, h_b\} = |FB|$$

and the inequalities hold, as claimed.

b) $|AC| \ge |BC|$. In this case the proof is similar.

The proof is complete.

A–I–4

The given equation can be rewritten as

$$\begin{gathered} k^2(l^2 + n^2) = 2015 + m^2(l^2 + n^2) \\ (k^2 - m^2)(l^2 + n^2) = 2015 \end{gathered}$$

We have $2015 = 5 \cdot 13 \cdot 31$ with primes 5, 13 and 31 and

$$5 \equiv 1 \pmod 4, \quad 13 \equiv 1 \pmod 4, \quad 31 \equiv 3 \pmod 4.$$

Since the sum of two square numbers is never congruent 3 modulo 4, and $l^2 + n^2 > 1$ holds, it follows that $l^2 + n^2$ is not a multiple of 31, and

$$l^2 + n^2 \in \{5, 13, 65\}.$$

a) $l^2 + n^2 = 5$. The only possible representation of 5 as a sum of two square numbers is $5 = 1^2 + 2^2$. It follows immediately that $\{l, n\} = \{1, 2\}$ and $l + n = 3$. This means that

$$k^2 - m^2 = (k + m)(k - m) = 13 \cdot 31.$$

Since $k + m > k - m$ we get

$$k + m = 31, \quad k - m = 13 \quad \text{or} \quad k + m = 13 \cdot 31 = 403, \quad k - m = 1.$$

In both cases, k and m are positive integers, so $(l + n) + (k + m)$ can be equal to $3 + 31 = 34$ or equal to $3 + 403 = 406$.

b) $l^2 + n^2 = 13$. It follows that $l^2 + n^2 = 3^2 + 2^2$, $l + n = 5$ and, consequently $k^2 - m^2 = (k + m)(k - m) = 5 \cdot 31$. This implies that

$$k + m = 31, \quad k - m = 5 \quad \text{or} \quad k + m = 5 \cdot 31 = 155, \quad k - m = 1.$$

Again, k and n are positive integers, so $(l + n) + (k + m)$ can attain the values $5 + 31 = 36$ or $5 + 155 = 160$.

c) $l^2 + n^2 = 65$. We must deal with two subcases: $l^2 + n^2 = 8^2 + 1^2$ or $l^2 + n^2 = 7^2 + 4^2$, so we have $l + n = 8 + 1 = 9$ or $l + n = 7 + 4 = 11$. It follows that $k^2 - m^2 = (k + m)(k - m) = 31$ and

$$k + m = 31, \quad k - m = 1.$$

Since k and n are positive integers, $(l + n) + (k + m)$ can attain the values $9 + 31 = 40$ and $11 + 31 = 42$.

This means that the set of possible values of $k + l + m + n$ is $\{34, 36, 40, 42, 160, 406\}$.

Category A (Team Competition)

A–T–1

We can rewrite the equation in the form

$$(x-2y)^2 = 3x + 2y - 4.$$

Since both x and y are integers, there exists an integer d such that

$$x - 2y = d,$$
$$3x + 2y - 4 = d^2.$$

Solving this system we obtain

$$x = \frac{d(d+1)}{4} + 1,$$
$$y = \frac{d(d-3)+4}{8}.$$

We check all residues modulo 8 to obtain that x and y are integers iff $d = 8k + 4$ ($k \in \mathbb{Z}$) or $d = 8(k-1) + 7 = 8k - 1$ ($k \in \mathbb{Z}$). In the fist case we have

$$x = 16k^2 + 18k + 6, \quad y = 8k^2 + 5k + 1, \quad k \in \mathbb{Z}, \tag{1}$$

the second case gives

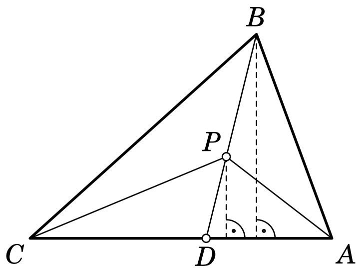
$$x = 16k^2 - 2k + 1, \quad y = 8k^2 - 5k + 1. \quad k \in \mathbb{Z}, \tag{2}$$

Conclusion. All integer solutions of the given equation are in the form (1) or (2).

A–T–2

First, note that if the point D divides a segment AC in ratio $u : v$, then for every point P on BD we have

$$\frac{S_{ABP}}{S_{BCP}} = u : v.$$



This results easily from the property that if $a : b = c : d$, then $a : b = (a + c) : (b + d) = (a - c) : (b - d)$. Indeed, the triangles ABD and BCD have a common altitude, as do the triangles ADP and CDP. Thus, the ratio of their areas is equal to

$$\frac{S_{ABD}}{S_{BCD}} = \frac{S_{ADP}}{S_{CDP}} = |AD| : |DC| = \frac{u}{v}.$$

Hence,

$$\frac{S_{ABP}}{S_{BCP}} = \frac{S_{ABD} - S_{ADP}}{S_{BCD} - S_{CDP}} = \frac{u}{v}.$$

Now, denote the ratio $|BP| : |PD|$ by $x : y$. We have

$$\frac{S_{ADP}}{S_{ABP}} = y : x \quad \text{and} \quad \frac{S_{CDP}}{S_{BCP}} = y : x$$

Now, we compute the ratio

$$\frac{S_{ACP}}{S_{BCP}} = \frac{S_{ADP} + S_{CDP}}{S_{BCP}} = \frac{S_{ADP}}{S_{BCP}} + \frac{S_{CDP}}{S_{BCP}} =$$

$$\frac{S_{ADP}}{S_{ABP}} \cdot \frac{S_{ABP}}{S_{BCP}} + \frac{y}{x} =$$

$$\frac{y}{x} \cdot \frac{u}{v} + \frac{y}{x} = \frac{y}{x} \cdot \left(\frac{u + v}{v}\right).$$

We want to choose P such that ratio is equal to $w : v$, so it is enough to put $y = w/v$, $x = v/(u + v)$ to achieve the required relationship.

A–T–3

We first note that $m = 0$ must hold if any of the numbers are equal to 0, since $m = abcdef$. In the following, we can therefore assume $m \neq 0$.

Since $a = bfm$ and $b = amc$ hold, multiplying both equations yields $cf = \frac{1}{m^2}$. Similarly we obtain $ad = \frac{1}{m^2}$ and $be = \frac{1}{m^2}$, and multiplying these three equations gives us

$$m = abcdef = \frac{1}{m^6}.$$

We therefore have $m^7 = 1 \iff m = 1$, and see that the only possible values for m are 0 or 1.

Finally, we must note that these values are both indeed possible, since all circled real numbers can equal 0 or 1, fulfilling the required properties.

Category B (Individual Competition)

B–I–1

Note that $45\,000 = 2^3 \cdot 3^2 \cdot 5^4$. The positive integer solutions (a, b, c) of the given equation must be of the form

$$a = 2^{\alpha_1} \cdot 3^{\beta_1} \cdot 5^{\gamma_1},$$
$$b = 2^{\alpha_2} \cdot 3^{\beta_2} \cdot 5^{\gamma_2},$$
$$c = 2^{\alpha_3} \cdot 3^{\beta_3} \cdot 5^{\gamma_3},$$

where exponents $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, 3)$ are non-negative integers fulfilling the system of equations

$$\alpha_1 + \alpha_2 + \alpha_3 = 3, \tag{1}$$
$$\beta_1 + \beta_2 + \beta_3 = 2, \tag{2}$$
$$\gamma_1 + \gamma_2 + \gamma_3 = 4. \tag{3}$$

The equation (1) is satisfied by triples $(\alpha_1, \alpha_2, \alpha_3)$ from the set $\{(3,0,0), (0,3,0), (0,0,3), (2,1,0), (1,2,0), (2,0,1), (1,0,2), (0,1,2), (0,2,1), (1,1,1)\}$, i.e. 10 triples altogether.

The equation (2) is satisfied by triples $(\beta_1, \beta_2, \beta_3)$ from the set $\{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}$, i.e. 6 triples altogether.

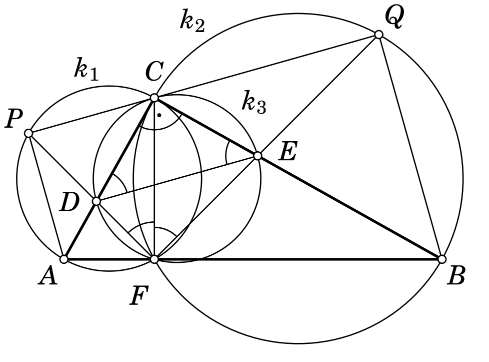
The last equation (3) is fulfilled by the triples $(\gamma_1, \gamma_2, \gamma_3)$ from the set $\{(4,0,0), (0,4,0), (0,0,4), (3,1,0), (1,3,0), (3,0,1), (1,0,3), (0,1,3), (0,3,1), (2,2,0), (2,0,2), (0,2,2), (2,1,1), (1,2,1), (1,1,2)\}$, i.e. 15 triples altogether.

It follows that the number of all triples $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, $(\gamma_1, \gamma_2, \gamma_3)$ fulfilling (1), (2) and (3) is equal, by the combinatorial product principle, to $10 \cdot 6 \cdot 15 = 900$.

Conclusion. There exist altogether 900 different positive integer solutions (a, b, c) of the given equation.

B–I–2

Since AP and AF are perpendicular to CP and CF, respectively, P and F are points of the circle k_1 with diameter AC. Triangle ACP is drawn external to ABC, so $AFCP$ is a convex cyclic quadrilateral with circumcircle k_1. Consequently, since ACP is an equilateral right triangle, we have $|\angle CFP| = |\angle CAP| = 45°$. Analogously, $BQCF$ is a cyclic quadrilateral with circumcircle k_2, and $|\angle QFC| = |\angle QBC| = 45°$.



This means that $|\angle QFP| = |\angle EFD| = 90° = |\angle DCE|$, so $CDEF$ is also a cyclic quadrilateral with DE as a diagonal. So finally we have $|\angle EDC| = |\angle EFC| = 45° = |\angle CFD| = |\angle CED|$, whence CDE is an isosceles right triangle, and $|DC| = |EC|$.

B–I–3

We can easily rewrite the given inequality in the form

$$3p + 3q + 3r + 3s - 4p - 4q - 4r \geq 0,$$

which is equivalent to the following inequality

$$3s - p - q - r = (s - p) + (s - q) + (s - r) \geq 0.$$

By the assumption we have $s - p \geq 0$, $s - q \geq 0$ and $s - r \geq 0$, the last inequality is true and the proof is finished.

Equality holds, iff $s - p = 0$, $s - q = 0$ and $s - r = 0$, i.e. iff $0 \leq p = q = r = s$.

B–I–4

Since $ABCD$ is a cimcumscribed quadrilateral with right angles at the vertices B and D with $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$, then following equalities hold

$$a^2 + b^2 = c^2 + d^2, \tag{1}$$

$$a + c = b + d. \tag{2}$$

We can rewrite (2) in the form

$$a - b = d - c$$

After squaring the last equation and using (1), easy algebraic manipulation gives us

$$2ab = 2cd. \tag{3}$$

Summing up (1) with (3) we consequently get $a+b = c+d$. Combining this result with (2) we further get $(a = d) \wedge (b = c)$. It means that $ABCD$ is a deltoid and the proof is finished.

Category B (Team Competition)

B–T–1

The following inequalities must necessarily hold for the numerators and denominators of the five given fractions

$$
\begin{aligned}
a+b &\geq c+d,\\
b+c &\geq d+e,\\
c+d &\geq e+a,\\
d+e &\geq a+b,\\
e+a &\geq b+c.
\end{aligned}
$$

Adding up all five inequalities we obtain

$$a+b+c+d+e \geq a+b+c+d+e.$$

Since equality in the last inequality holds, it must also hold in all five inequalities equalities, i.e.

$$
\begin{aligned}
a+b &= c+d, && (1)\\
b+c &= d+e, && (2)\\
c+d &= e+a, && (3)\\
d+e &= a+b, && (4)\\
e+a &= b+c. && (5)
\end{aligned}
$$

From (1), (3) and (5) we have $a=c=e$. From (2) we further get $b=d$, and finally (5) implies $a=b=c=d=e$.

Conclusion. After *obligatory* checking we can see that all desired 5-tuples (a,b,c,d,e) of positive integers are 5-tuples in the form (n,n,n,n,n), where n is an arbitrary positive integer.

B–T–2

a) If the sticks are of length 1, 1, 2, 3, there is no way to form a triangle, since the triangle inequality cannot hold. This yields a circumference of 7. If the circumference is less than 7, there are either 3 sticks of length 1 (yielding an equilateral triangle) or two of length 2, which can be sides of an isosceles triangle together with a stick of length 1.

b) If the sticks are of lengths 1, 1, 2, 3, 5 and 8, there is no way to form a triangle. (The Fibonacci numbers never fulfill the triangle inequalities.) If the longest stick is less than 8 units in length, the next larger ones must be either 3 and 4 in length (meaning that they form a triangle with a stick of length 2 or there are 3 sticks of length 1, forming an equilateral triangle), or 3 and 3 in length (yielding an isosceles triangle with a stick of shorter length), or there are again 4 sticks, the sum of whose lengths is less than 7, which yields a triangle as seen in part a).

B–T–3

We have

$$\overline{abccba} = 100001a + 10010b + 1100c = 7(14286a + 1430b + 157c) - (a - c).$$

Such number is divisible by 7 iff $(a-c)$ is divisible by 7. $a \neq 0$ and c are digits, therefore $-8 \le a - c \le 9$. This follows $(a-c) \in \{-7, 0, 7\}$. For $a - c = -7$ we have

$$(a, c) \in \{(1, 8), (2, 9)\},$$

for $a - c = 0$ we have

$$(a, c) \in \{(1, 1), (2, 2), \dots, (9, 9)\}$$

and finally for $a - c = 7$ we have

$$(a, c) \in \{(7, 0), (8, 1), (9, 2)\},$$

altogether 14 possibilities for (a, c). In all cases b is arbitrary digit, therefore there exists $14 \cdot 10 = 140$ six-digit palindromes which are divisible by 7.

Category C (Individual Competition)

C–I–1

Firstly, we can see that $m \neq 0 \neq n$. Further, we can rewrite the given equation in the form

$$(m-n)+\left(\frac{1}{n}-\frac{1}{m}\right)=(m-n)\left(1+\frac{1}{mn}\right)=0.$$

This implies either $m-n=0$, i.e. $m=n$, or $1+\frac{1}{mn}=0$, i.e. $mn=-1$.

Conclusion. The first case gives solutions of the given problem in the form $(m,n)=(n,n)$, where n is any positive integer. The second case yields another two solutions, $(m,n)=(1;-1)$ and $(m,n)=(-1;1)$.

C–I–2

Since the sum of the internal and external angles must be 180°, we have $\alpha+\varepsilon=180^\circ$. This yields

$$180^\circ=\alpha+k\alpha=(1+k)\alpha.$$

We see that $1+k$ must be a divisor of 180. The set of all divisors of 180 is

$$\{1,2,3,4,5,6,9,10,12,15,18,20,30,36,45,60,90,180\}.$$

We note that $k=0$ is obviously not possible. $k=1$ means $\alpha=\varepsilon=90^\circ$, and since the triangle is acute-angled, this is also not possible. Furthermore, $k=179$ yields $\alpha=1^\circ$, and therefore $\beta+\gamma=179^\circ$. This means that one of the angles β or γ is equal to or greater than 90°, which is also not possible. These arguments eliminate the values 1, 2 and 180 from the list of possible values of $1+k$. All other values of $1+k$ are possible, and the set of all possible values for k is therefore

$$\{2,3,4,5,8,9,11,14,17,19,29,35,44,59,89\}.$$

C–I–3

All desired triples of positive integers a, b, c must necessarily fulfill the inequalities

$$a + b \geq b + c, \tag{1}$$

$$b + c \geq c + a, \tag{2}$$

$$c + a \geq a + b. \tag{3}$$

From (1)–(3)

$$b \geq a \geq c \geq b$$

immediately follows and thus $a = b = c$.

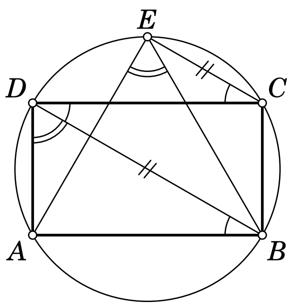
Conclusion. After *obligatory* checking we can see that all desired triples (a, b, c) of positive integers are the triples (m, m, m), where m is any positive integer.

C–I–4

We begin by noting that triangles ABD and BAC are congruent halves of the rectangle $ABCD$, and we therefore have

$$|\angle ACB| = |\angle ADB| = 180^\circ - 90^\circ - 30^\circ = 60^\circ.$$

Since points A, B, C, D, E lie on a common circle, we therefore have $|\angle AEB| = |\angle ADB| = 60^\circ$.



Furthermore, we have $|\angle CDB| = |\angle ECD| = |\angle DBA| = 30^\circ$, and therefore $|\angle ABE| = |\angle ABD| + |\angle DBE| = 30^\circ + 30^\circ = 60^\circ$. We see that triangle ABE is equilateral, as it has two interior angles of 60°, and we therefore have $|AE| = |AB| = 4$.

Category C (Team Competition)

C–T–1

The inequality implies $|3y| < 23$, and thus $|y| \in \{0,1,2,3,4,5,6,7\}$. We discuss all such cases in the following table.

$\|y\|$	$\|x\| \in$	# of y	# of x	# of pairs (x,y)
0	$\{0,1,2,\ldots,11\}$	1	23	23
1	$\{0,1,2,\ldots,9\}$	2	19	38
2	$\{0,1,2,\ldots,8\}$	2	17	34
3	$\{0,1,2,\ldots,6\}$	2	13	26
4	$\{0,1,2,\ldots,5\}$	2	11	22
5	$\{0,1,2,3\}$	2	7	14
6	$\{0,1,2\}$	2	5	10
7	$\{0\}$	2	1	2
Total # of pairs				169

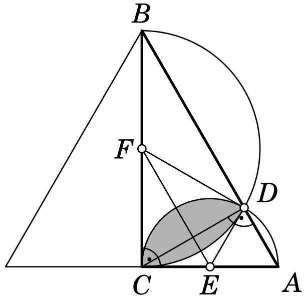
There are 169 pairs of integers satisfying the inequality

$$2|x| + 3|y| < 23.$$

C–T–2

By the Pythagorean theorem we can easy calculate the length of the third side of the triangle ABC. It is equal to 2. We can therefore state that the triangle ABC is half of an equilateral triangle whose side is equal to 2. Then the angles $\angle ABC$ and $\angle BAC$ are equal to 30°

and 60°, respectively.



Now, denote by D the foot of the altitude of the triangle ABC drawn from the vertex C. Since the angles $\angle ADC$ and $\angle BDC$ are both right angles, the point D, by the converse of the Thales' theorem, lies on both circles and so it is the second common point (the first is C). Denote further E and F as midpoints of AC and BC.

Thus, the common part of the circles is the common part of two circle sectors contained in the triangle ABC with central angles $\angle CED$ and $\angle CFD$ and radii $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, respectively. By the inscribed angle theorem we have

$$|\angle CED| = 2\,|\angle CAD| = 120^\circ, \quad |\angle CFD| = 2\,|\angle CBD| = 60^\circ.$$

The area S of a common part of the circle sectors can be calculated by the formula $S = S_E + S_F - S_Q$, where S_E and S_F are areas of the circle sectors with central angles at E and F, respectively, and S_Q is the area of the quadrilateral $CEDF$.

By the well known formulas for the area of a circle sector, we have

$$S_E = \frac{120^\circ}{360^\circ}\pi\left(\frac{1}{2}\right)^2 = \frac{1}{12}\pi, \qquad S_F = \frac{60^\circ}{360^\circ}\pi\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{8}\pi.$$

The area S_Q of quadrilateral $CEDF$ is equal to

$$S_Q = 2S_{CEF} = 2\cdot\frac{1}{4}S_{ABC} = \frac{\sqrt{3}}{4}.$$

Therefore, we have

$$S = \frac{1}{12}\pi + \frac{1}{8}\pi - \frac{\sqrt{3}}{4} = \frac{5}{24}\pi.$$

C–T–3

a) The smallest possible sum is given by the expression

$$20659 + 14387 = 35046.$$

b) The largest such number is 9785634120.

c) There are many such combinations. Examples are

$$120 \cdot 142 + 231 = 17271 \quad \text{or} \quad 101 \cdot 101 + 101 = 10302.$$

Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Marta Mościcka	V LO Bielsko-Biała	8	5	8	8	**29**
2.	Wojciech Klemens	V LO Bielsko-Biała	3	8	8	8	**27**
3.	Marian Poljak	GJŠ Přerov	8	1	8	8	**25**
	Petr Vincena	GJŠ Přerov	8	1	8	8	**25**
5.	Jakub Paliga	AZSO Chorzów	8	0	8	8	**24**
6.	Jakub Kuklis	V LO Bielsko-Biała	3	7	8	5	**23**
7.	Tomáš Kremel	GJŠ Přerov	8	0	8	6	**22**
8.	Jan Gocník	GJŠ Přerov	8	0	0	4	**12**
9.	Michał Ślusarczyk	AZSO Chorzów	3	0	8	0	**11**
	Wojciech Buś	V LO Bielsko-Biała	6	3	0	2	**11**
11.	Gerda Prach	BRG Kepler	6	0	0	4	**10**
12.	Vladimír Jeřábek	GMK Bílovec	8	0	0	1	**9**
13.	Sebastian Borówka	AZSO Chorzów	4	0	0	4	**8**
14.	Lucie Holušová	GMK Bílovec	4	0	0	2	**6**
15.	Tomáš Moravec	GMK Bílovec	0	0	0	4	**4**
16.	Benedikt Andritsch	BRG Kepler	3	0	0	0	**3**
17.	Artur Wyciślok	AZSO Chorzów	1	0	0	1	**2**
	Martin Ošťádal	GMK Bílovec	1	0	0	1	**2**
19.	Markus Ruprechter	BRG Kepler	0	0	0	1	**1**
20.	Doris Prach	BRG Kepler	0	0	0	0	**0**

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Bogna Pawlus	V LO Bielsko-Biała	8	8	8	8	**32**
2.	Mikołaj Tomalik	V LO Bielsko-Biała	8	0	8	8	**24**
3.	Jiří Nábělek	GMK Bílovec	8	0	8	6	**22**
4.	Konstantin Andritsch	BRG Kepler	0	8	8	2	**18**
5.	Marcin Socha	AZSO Chorzów	0	0	8	8	**16**
	Bára Tížková	GMK Bílovec	0	2	6	8	**16**
7.	Jan Równicki	AZSO Chorzów	0	8	6	0	**14**
	Łukasz Grzesiek	V LO Bielsko-Biała	0	0	6	8	**14**
9.	Damian Wałoszek	GMK Bílovec	8	0	2	0	**10**
	Jan Šuta	GJŠ Přerov	0	1	8	1	**10**
11.	Denisa Chytilová	GJŠ Přerov	0	0	8	1	**9**
	Daniel Horiatakis	BRG Kepler	0	1	8	0	**9**
13.	Julie Dostalíková	GJŠ Přerov	0	0	8	0	**8**
	Adrian Steinmann	BRG Kepler	0	0	8	0	**8**
	Bazyli Polednia	V LO Bielsko-Biała	0	0	8	0	**8**
16.	Tereza Špalková	GJŠ Přerov	0	0	4	0	**4**
17.	Karol Szydlik	AZSO Chorzów	0	0	3	0	**3**
18.	Grzegorz Pielot	AZSO Chorzów	0	0	2	0	**2**
	Tomáš Kůřil	GMK Bílovec	0	0	2	0	**2**

Category C (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Tomáš Křižák	GMK Bílovec	4	7	8	8	**27**
2.	Michaela Svatošová	GMK Bílovec	2	8	8	8	**26**
3.	Michał Szwej	AZSO Chorzów	4	7	8	6	**25**
4.	Jana Pallová	GJŠ Přerov	6	0	8	8	**22**
5.	Marcin Traskowski	AZSO Chorzów	6	7	8	0	**21**
6.	Minh Nguyen	BRG Kepler	2	7	8	0	**17**
7.	Matyáš Florík	GMK Bílovec	4	0	4	8	**16**
8.	Jaroslav Hradil	GJŠ Přerov	0	1	0	8	**9**
9.	Krzysztof Stefan	AZSO Chorzów	2	0	4	0	**6**
10.	Julian Narimany	AZSO Chorzów	2	0	2	0	**4**
	Dominik Nagy	GMK Bílovec	4	0	0	0	**4**
	Vít Horčička	GJŠ Přerov	2	0	2	0	**4**
13.	Jakub Gogela	GJŠ Přerov	2	0	0	0	**2**
	Julian Wonisch	BRG Kepler	2	0	0	0	**2**

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	V LO Bielsko-Biała	4	8	8	**20**
2.	GJŠ Přerov	2	8	8	**18**
3.	BRG Kepler	1	6	8	**15**
4.	AZSO Chorzów	0	7	2	**9**
5.	GMK Bílovec	0	0	6	**6**

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	V LO Bielsko-Biała	8	8	8	**24**
2.	AZSO Chorzów	6	4	8	**18**
	GMK Bílovec	6	4	8	**18**
4.	GJŠ Přerov	6	4	6	**16**
5.	BRG Kepler	6	4	4	**14**

Category C (Team Competition)

Rank	School	1	2	3	∑
1.	GMK Bílovec	8	8	8	**24**
2.	AZSO Chorzów	4	2	8	**14**
3.	GJŠ Přerov	2	0	8	**10**
	BRG Kepler	4	0	6	**10**

RNDr. Jaroslav Švrček, CSc.
RNDr. Pavel Calábek, Ph.D.
Dr. Robert Geretschläger
Dr. Gottfried Perz
Dr. Józef Kalinowski
Dr. Jacek Uryga

Mathematical Duel ’15

Určeno pro talentované žáky středních a základních škol
České republiky a posluchače matematických oborů
Přírodovědecké fakulty UP v Olomouci.

Výkonný redaktor prof. RNDr. Zdeněk Dvořák, DrSc.
Odpovědná redaktorka Mgr. Jana Kreiselová
Technické zpracování RNDr. Pavel Calábek, Ph.D.

Publikace neprošla ve vydavatelství redakční a jazykovou úpravou.

Vydala a vytiskla Univerzita Palackého v Olomouci
Křížkovského 8, 771 47 Olomouc
www.upol.cz/vup
e-mail: vup@upol.cz

Olomouc 2015

1. vydání

Neprodejné

ISBN 978-80-244-4650-9